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Automorphic Decompositions of Graphs

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AUTOMORPHIC DECOMPOSITIONS OF GRAPHS

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ABSTRACT

Let G and H be graphs. A G-decomposition $\mathscr{D}$ of a graph H is a partition of the edge set of H such that the subgraph induced by the edges in each part of the partition is isomorphic to G. It is well known that a graceful labelling (or more generally a ρ-valuation) of a graph G induces a cyclic G-decomposition of a complete graph. We will extend these notions to that of a general valuation in a cyclic group. Such valuations yield decompositions of circulant graphs. We will show that every graph has a valuation and hence is a divisor of arbitrarily large circulant graphs. The problem of which graphs can host a decomposition by a graph G is much more difficult and is still open.

Let $\mathscr{D}$ be a G-decomposition of H. The intersection graph generated by the decomposition $\mathscr{D}$, denoted $I(\mathscr{D})$, has a vertex for each part of the partition and an edge if the parts of the partition share a common node in H. While the problem of determining whether a G-decomposition of H exists is a well-studied one, the structure of the resulting intersection graph has not been the subject of much research. As such, we are interested in the structure of these graphs. Of specific interest is whether there exist H, G, and a G-decomposition of H such that the resulting intersection graph is isomorphic to H. A decomposition $\mathscr{D}$ such that $I(\mathscr{D}) \cong H$ is said to be automorphic ("self-shaped"). If a graph H is able to host an automorphic decomposition, then we say that H is an automorphic host. Similarly, if there exists an H such that there exists an automorphic G-decomposition of H, then we say that G is an automorphic divisor.

In this dissertation, we will give several necessary conditions for the existence of an automorphic G-decomposition of H. These will allow us to examine the problem of which graphs are automorphic hosts. We conjecture that only even regular graphs are able to host an automorphic decomposition. In an attempt to prove this conjecture, we have shown the following special cases:

- If $\chi(H) \leq 3$, then H is $2e(G)$-regular.
- If $\chi(H) = n(G)$ and G is d-regular, then H is $n(G)d$-regular.
- If $\chi(H) = n(G)$, G is not a disjoint union of P_2's, and the smallest elements of $dseq(G)$ are 1 and a, where $a \geq 2e(G) - n(G) + 1$, then H must be $2e(G)$-regular.
- If $G \cong P_4$ and $\chi(H) = 4$, then H is 6-regular.
- If G is d-regular and any two blocks share at most one common node in H, then H is $n(G)d$-regular.

This dissertation will also examine the problem of which graphs are automorphic divisors. We conjecture that every graph is an automorphic divisor. In an attempt to prove this, we generalize our notion of a valuation for arbitrary groups. This allows us to consider direct products of the related valuations.

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CHAPTER 1
BASIC DEFINITIONS

The goal of this chapter is to introduce the major ideas that will be used throughout the dissertation. Readers who are familiar with the standard texts will recognize many of these terms. As notation often varies in different sources, we wish to standardize the internal notation of this document.

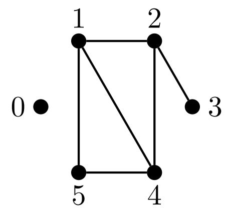
1.1 Basic Graph Theory

A *(simple) graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and an incidence relation that associates with each edge exactly two vertices called its *endpoints*. Often, we will use the more compact notation $G = (V, E)$. In this case the incidence relation will be implicit. An edge e with endpoints $u, v \in V(G)$ will often be denoted uv. If $uv \in E(G)$, then we say that u and v are *adjacent* and are *neighbors*. We say that an edge e is *incident* with a vertex v if and only if e has v as one of its endpoints. A graph $G = (V, E)$ is *finite* if and only if V and E are both finite sets [75]. Unless otherwise explicitly stated, we will assume that all graphs in this dissertation are finite.

Let G and H be graphs. We say that G is a *subgraph*, or more properly, an *edge induced subgraph*, of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. In this case, we say that H *contains* G and write $G \subseteq H$ [75]. Let $A \subseteq V(G)$. The *subgraph of G induced by A*, denoted G_A, is the graph with vertex set A and $E(G_A) = \{xy : x, y \in A, xy \in E(G)\}$. Such graphs are called *vertex induced subgraphs* [33].

An *isomorphism* between two graphs G and H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If such an isomorphism exists, we say that G is *isomorphic* to H, and denote this by $G \cong H$. We say that H is *G-free* if for all $A \subseteq V(H)$, H_A is not isomorphic to G. If H is K_3-free, we often say that H is *triangle-free* [75].

Figure 1.1 A Graph



The *degree* of a vertex v in G, denoted $deg_G(v)$, is the number of edges in G that contain v as an endpoint [9]. We say that $v \in V(G)$ is an *isolated vertex* if and only if $deg_G(v) = 0$. We say that $v \in V(G)$ is a *pendant vertex* if and only if $deg_G(v) = 1$. An edge of G is a *pendant edge* if and only if it is incident with a pendant vertex. We say that $v \in V(G)$ is a *universal vertex* if it is adjacent to every other vertex of G.

- The *order* of a graph G, denoted $n(G)$, is defined as $n(G) := |V(G)|$.
- The *size* of a graph G, denoted $e(G)$, is defined as $e(G) := |E(G)|$.
- The *minimum degree* of G, denoted $\delta(G)$, is defined as $\delta(G) := \min\{deg_G(v) : v \in V(G)\}$.
- The *maximum degree* of G, denoted $\Delta(G)$, is defined as $\Delta(G) := \max\{deg_G(v) : v \in V(G)\}$.
- The *average degree* of G, denoted $\bar{d}(G)$, is defined as:

$$\bar{d}(G) := \frac{1}{n(G)} \sum_{v \in V(G)} deg_G(v).$$

Example 1.1.1 *Consider the graph G given in Figure 1.1. Here, $V(G) = \{0, 1, ..., 5\}$ and $E(G) = \{12, 14, 15, 23, 24, 45\}$. In this graph, the vertex labelled 0 is an isolated vertex, while*

3 is a pendant vertex. The edge 23 is a pendant edge. We also have that $\delta(G) = 0$ *and* $\Delta(G) = 3$.

Proposition 1.1.2 *[9, 75] For a graph* G*, we have the following:*

(i) $\sum_{v \in V(G)} deg_G(v) = 2e(G)$.

(ii) $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

(iii) $\bar{d}(G) = \frac{2e(G)}{n(G)}$.

(iv) $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.

(v) $e(G) = \frac{\bar{d}(G)n(G)}{2}$.

(vi) $\frac{\delta(G)n(G)}{2} \leq e(G) \leq \frac{\Delta(G)n(G)}{2}$.

Note that West [75] does not use the average degree of a graph. However, the results of Proposition 1.1.2 follow readily from statements in West as well as the above definitions.

A *clique* in a graph is a set of pairwise adjacent vertices. The *clique number* of a graph G, denoted $\omega(G)$, is the cardinality of a largest clique in G. An *independent set* in a graph is a set of pairwise non-adjacent vertices. The *independence number* of a graph G, denoted $\alpha(G)$, is the cardinality of a largest independent set. A graph G is k*-partite* if $V(G)$ can be expressed as the disjoint union of k independent sets. In particular, G is *bipartite* if $V(G)$ can be expressed as the disjoint union of two independent sets. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k for which G is a k-partite graph.

> *When discussing structure of graphs, it is convenient to have names and notation for important isomorphism classes. We want the flexibility to refer to the isomorphism class or to any representative of it.* — West [75], p. 9

We now list several important isomorphism classes of graphs:

(i) A *walk* is an ordered list of vertices such that if two vertices appear consecutively on the list, they are adjacent.

(ii) A *path* is a walk such that no vertex appears twice on the list. A path on n vertices will be denoted P_n.

(iii) A *cycle* is a walk such that the first and last vertex on the list are the same and no other vertex is repeated. A cycle on n vertices will be denoted C_n.

(iv) A *complete graph* is a graph where any two distinct vertices are adjacent. A complete graph on n vertices is denoted K_n.

(v) A *complete bipartite graph* is a graph whose vertex set can be partitioned into two parts such that two vertices are adjacent if and only if they are in different parts. When the parts contain n and m vertices, we denote this graph by $K_{n,m}$.

(vi) In particular, $K_{1,n}$ is called a *star*.

(vii) An *n-dimensional hypercube*, denoted Q_n, is a graph whose vertex set is the set of all n-tuples with entries in $\{0,1\}$ and whose edges are the pairs of n-tuples which differ in exactly one position.

(viii) The *Petersen Graph* is the graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.

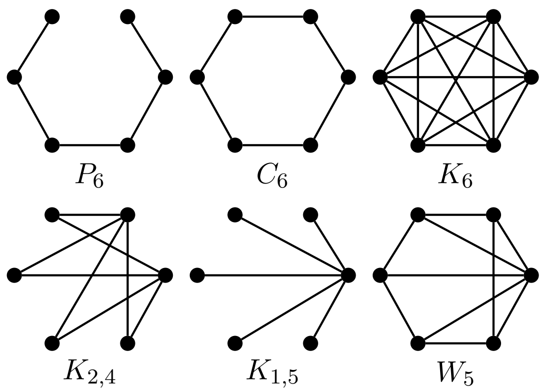
(ix) The *Caterpillar* $C(a_1, ..., a_n)$ is the graph obtained from P_n by adjoining a_i pendant edges to the *ith* vertex of the path.

(x) The *wheel*, denoted W_n, is the graph obtained from C_n by adjoining a universal vertex to C_n.

Figure 1.2 shows P_6, C_6, K_6, $K_{2,4}$, $K_{1,5}$ and W_5. An example of a caterpillar is given in Figure 2.1. An example of Q_3 can be found in Figure 2.6.

G is *connected* if for any two distinct vertices $u, v \in V(G)$, G contains a path which has as its endpoints u and v. A *tree* is a connected graph that contains no cycle. A graph G is said to be *k-regular* if $\delta(G) = k = \Delta(G)$. In particular, if G is k-regular for some even number k, then we say that G is *even regular*.

Figure 1.2 Special Graphs on Six Vertices



Proposition 1.1.3 *A connected graph is* 2*-regular if and only if it is a cycle.*

Proof.

If G is a cycle, then it is clearly a 2-regular connected graph. Conversely, suppose that G is a 2-regular connected graph. Let P be a path of maximal length in G. Let u and v be the endpoints of P. Note that $deg_P(u) = 1$ by definition but $deg_G(u) = 2$ by hypothesis. Hence, there exists $w \in V(G)$ such that $uw \in E(G)$, but $uw \notin E(P)$. If $w \notin V(P)$, then the path from v to w is strictly longer than that of P. This contradicts P being a path of maximal length in G. Thus, we may assume that $w \in P$. Note that $deg_P(w) = 2$ by definition. If $w \neq v$, then u and w are adjacent and $uw \notin E(P)$. Hence, $deg_G(w) = 3$ which contradicts the assumption that G is 2-regular. Thus $w = v$. Further, P with the edge uw is a cycle C. Note that no vertex in C is adjacent to a vertex not in C because G is a 2-regular graph. If there were a vertex of G not in C, then G would be disconnected, contrary to hypothesis. Thus, $G = C$, so G is a cycle. ■

Proposition 1.1.4 *(Bounds on the chromatic number [75])*

(i) $\chi(G) \geq \omega(G)$.

(ii) $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

(iii) $\chi(G) \leq \Delta(G) + 1$.

(iv) *If* G *is a connected graph other than a complete graph or an odd cycle, then* $\chi(G) \leq \Delta(G)$.

Remark 1.1.5 *Items (3) and (4) are often referred to as Brooks' Theorem [10].*

Proposition 1.1.6 *Let* G *be a graph.*

(i) If G *is a tree, then* $e(G) = n(G) - 1$ *[75].*

(ii) If G *is connected, then* $e(G) \geq n(G) - 1$ *[75].*

(iii) If G *has no isolated vertices, then* $e(G) \geq \frac{n(G)}{2}$.

Proof.

(i) We proceed by induction on $n = n(G)$. If $n = 1$, an acyclic graph with one vertex has zero edges. Thus, we may assume for some $n \geq 1$ that $e(G) = n(G) - 1$. If G has $n + 1$ vertices, we may delete one pendant vertex and its corresponding edge from G. Note that the resulting graph G' has order n and is acyclic and connected. Applying the inductive hypothesis to G' yields $e(G') = n(G') - 1$. Observing that $e(G) = e(G') + 1$ and $n(G) = n(G') + 1$ yields the desired result.

(ii) Suppose that G is connected. If G is a tree, then $e(G) = n(G) - 1$ and we are done. If G is connected and contains a cycle, then we may delete one edge from that cycle and retain connectivity. Thus, by deleting one edge from each cycle in G, we retain connectivity and the resulting graph G' is a tree. Since $e(G') = n(G') - 1 = n(G) - 1$ and G has at least as many edges as G', we have $e(G) \geq n(G) - 1$.

(iii) If G has no isolated vertices, then $\delta(G) \geq 1$. By Proposition 1.1.2, we have:

$$e(G) \geq \frac{n(G)\delta(G)}{2} \geq \frac{n(G)}{2}.$$

■

The *disjoint union* of two disjoint simple graphs G and H, denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ [75]. The *join* of two simple graphs G and H, written as $G \vee H$, is the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = \{xy : x \in V(G), y \in V(H)\} \cup E(G) \cup E(H)$ [75]. The *complement* of a graph G, denoted $\bar{G}$, is defined as the graph with $V(\bar{G}) = V(G)$ and $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$ [75].

Proposition 1.1.7 *Let G and H be graphs. Then $\overline{G \cup H} \cong \overline{G} \vee \overline{H}$.*

Proof.

Note that $V(\overline{G \cup H}) = V(\overline{G} \vee \overline{H})$. We need only show that the two graphs have the same edge set. Let $uv \in E(\overline{G} \vee \overline{H})$. Hence, we have one of three exclusive possibilities, $uv \in E(\overline{G})$, $uv \in E(\overline{H})$, or uv is generated by the join of the two graphs. If $uv \in E(\overline{G})$, then $uv \notin E(G)$. Thus, $uv \notin E(G \cup H)$ or equivalently, $uv \in E(\overline{G \cup H})$. A similar argument holds if $uv \in E(\overline{H})$. If uv is generated by the join, then $uv \notin E(G \cup H)$, or equivalently, $uv \in E(\overline{G \cup H})$.

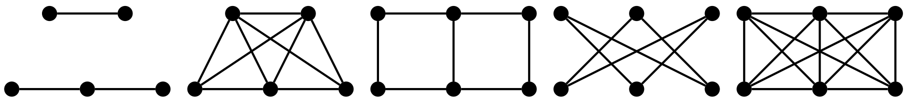
Conversely, if $uv \notin E(\overline{G} \vee \overline{H})$, then $uv \notin E(\overline{G})$ and $uv \notin E(\overline{H})$. If $u \in V(\overline{G})$ and $v \in V(\overline{H})$, then $uv \in E(\overline{G} \vee \overline{H})$ by the definition of join. A similar argument holds for the case where $u \in V(\overline{H})$ and $v \in V(\overline{G})$. Hence, $uv \in E(G)$ or $uv \in E(H)$. Thus, by the definition of union, we have $uv \in E(G \cup H)$, or equivalently, $uv \notin E(\overline{G \cup H})$. ■

Examples of products on graphs are as follows:

(i) The *Cartesian Product* of G and H, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where (u, v) is adjacent to (u', v') if and only if

1) $u = u'$ and $vv' \in E(H)$ or

Figure 1.3 Products on Graphs



2) $v = v'$ and $uu' \in E(G)$ [9, 75].

(ii) The *direct product* of G and H, written as $G \times H$, is the graph with vertex set $V(G) \times V(H)$ where (u, v) is adjacent to (u', v') if and only if $uu' \in E(G)$ and $vv' \in E(H)$ [9].

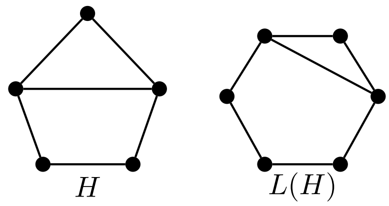
(iii) The *strong product* of G and H, written as $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ where (u, v) is adjacent to (u', v') if and only if they are adjacent in $G \square H$ or $G \times H$ [9].

Examples of these products, as well as the union and the join, are given in Figure 1.3.

The main problem of this dissertation is concerned with decompositions of graphs. Given graphs G and H, a G-*decomposition* of H is a partition of the edge set of H into isomorphic copies of G. In this case, we say that G is a *divisor* of H and denote this by $G|H$. The graph H is called the *host* of the decomposition. The graph G is called the *prototype* of the decomposition. The parts of the partition are called G-*blocks*, or more simply *blocks* [9]. It should be noted that if either our host or our prototype contains isolated vertices, then this would lead to a degenerate case.

Given graphs G and H, there are essentially three ways of showing the existence of a G-decomposition of H. The first is an ad hoc method, i.e., list the G-blocks in the decomposition and then confirm that this is the required decomposition. Second, one might use an algebraic method to construct the decomposition. The third method is by using a design argument. Decompositions arising from designs will be discussed in Section 1.4. Methods

Figure 1.4 A Graph and its Line Graph



involving algebraic constructions will be in Chapter Two and Chapter Five. However, we will give an introduction to pertinent abstract algebra in the next section.

We now come to a central notion of this dissertation. The *intersection graph* generated by a decomposition $\mathscr{D}$, denoted $I(\mathscr{D})$, has a vertex for each block in the decomposition and two vertices are adjacent in $I(\mathscr{D})$ if and only if the corresponding blocks share a common vertex in H. There are many different kinds of intersection graphs in the literature [61]. Thus, we will often refer to intersection graphs generated by a G-decomposition of a graph H as *G-decomposition graphs.*

Remark 1.1.8 *Because of the multiple levels of abstraction inherent in decomposition graphs, it is useful to make the following distinction. In discussing G-decomposition graphs, the elements of $V(H)$ and $V(G)$ will be referred to as* nodes*, while the elements of $V(I(\mathscr{D}))$ will be referred to as* vertices. *When discussing general graphs without regard to a decomposition, the term* vertex *will be used solely. This is the convention established by Jamison, Mendelsohn, and Mulder [23, 44, 45, 46, 47, 48, 49, 50, 52].*

Perhaps the most familiar type of G-decomposition graphs are called line graphs. The *line graph* of H, denoted $L(H)$, is the P_2-decomposition graph of H. An example of a line graph is given in Figure 1.4.

Proposition 1.1.9 (Line graphs of special graphs)

(i) The line graph of a connected graph is likewise connected.

(ii) $L(C_n) \cong C_n$.

(iii) $L(P_n) \cong P_{n-1}$.

(iv) $L(K_{1,n}) \cong K_n$.

Proof.

(i) Let H be connected and take $e, f \in E(H)$. If u, v are endpoints of e, f respectively, then there exists a path between u, v. Thus, there exists a path between the vertices representing e and f in $L(H)$. Ergo, $L(H)$ is connected by definition.

(ii) Since C_n is connected, it follows from (i) that $L(C_n)$ is connected. Note that each edge in C_n shares an endpoint with exactly two others. Thus, $L(C_n)$ is a 2-regular connected graph. Hence, it follows by Proposition 1.1.3 that $L(C_n) \cong C_n$.

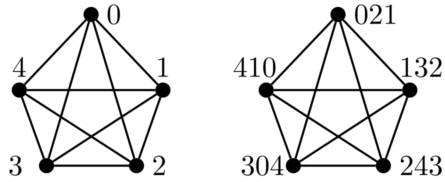
(iii) Note that P_n can be obtained from C_n by deleting a single edge. Thus, $L(P_n)$ can be obtained from $L(C_n)$ by deleting a node in C_n. Hence, $L(P_n) \cong P_{n-1}$.

(iv) Note that each of the edges share a common center. Thus, in the line graph, the vertices are mutually adjacent. Since $e(K_{1,n}) = n$, it follows that $L(K_{1,n}) \cong K_n$.

■

In the next chapter, we will examine the properties of G-decompositions and their corresponding intersection graphs in more detail. This will allow us to examine the main problem of this dissertation: Does there exist a G-decomposition of H, $\mathscr{D}$, such that $I(\mathscr{D}) \cong H$? Note that the line graph (i.e., a P_2-decomposition graph) of C_n is isomorphic to C_n by Proposition 1.1.9. A less trivial example of a P_3-decomposition where $I(\mathscr{D}) \cong H$ is given in Figure 1.5. The machinery necessary to show that this is the required decomposition will be given in Proposition 2.1.6, Theorem 2.3.10, and Proposition 2.4.6. Such decompositions can be considered "self-shaped" or literally, *automorphic*. If $\mathscr{D}$ is a G-decomposition of H, we say that $\mathscr{D}$ is an *automorphic G-decomposition* of H if $I(\mathscr{D}) \cong H$. The study of automorphic decompositions will be the primary focus of this dissertation.

Figure 1.5 An Automorphic P_3-Decomposition of K_5



In our study of automorphic decompositions, we introduce several refinements on the idea. In a *simple* G-decomposition of H, any two blocks in the decomposition share at most one common node in H. We will show in Proposition 2.3.6 that any K_p-decomposition of H is simple. Further, a P_2-decomposition of C_n is an example of a simple automorphic decomposition. We will show in Theorem 4.1.1 that if G is regular and a simple automorphic divisor of H, then H must be regular of even degree.

If $n(G) = \chi(H)$ and $\mathscr{D}$ is an automorphic G-decomposition of H, then $\mathscr{D}$ is a *fully automorphic G-decomposition* of H. If $H = C_{2n}$, then an example of a P_2-decomposition that is both simple and fully automorphic is the line graph. However, if $H = C_{2n+1}$, then this decomposition is simple automorphic, but not fully automorphic. The example given in Figure 1.5 is neither simple nor fully automorphic; it is, however, automorphic. Additional examples of fully automorphic decompositions will be given later.

In Chapter Three, we will give additional examples of automorphic decompositions. Often, these examples will rely on labelling the prototype G with elements of a cyclic group. Further, we will establish necessary conditions for the existence of an automorphic decomposition. These necessary conditions will often rely on number theoretical concepts. In order to facilitate these discussions, we present introductory information on abstract algebra and number theory in the next section.

1.2 A Brief Introduction to Abstract Algebra and Number Theory

We are often concerned with labelling the vertices of a graph such that these labels come from a particular group. Because of this, we give a very brief introduction to group theory. Note that although G is typically used to denote a group, this would create confusion, as we will be using G to denote a graph. As such, we will use Γ to denote a group. In general, if there is a conflict between standard notation in group theory and graph theory, we will use the standard notation in graph theory and use a semi-standard, or at least alternative notation for our groups. As group theory will only be used sporadically throughout this document, this will keep the different levels of abstraction sufficiently clear.

Definition 1.2.1 *[40] A* group $(\Gamma, *)$ *is a non-empty set* Γ *with a binary operation* $*$ *that satisfies the following:*

(i) Closure: *If* $a, b \in \Gamma$*, then* $a * b \in \Gamma$*.*

(ii) Associativity: *If* $a, b, c \in \Gamma$*, then* $(a * b) * c = a * (b * c)$*.*

(iii) Existence of an Identity: *There exists* $e \in \Gamma$ *such that for all* $a \in \Gamma$*,* $a * e = e * a = a$*.*

(iv) Existence of Inverses: *For all* $a \in \Gamma$ *there exists* $a^{-1} \in \Gamma$ *such that* $a*a^{-1} = a^{-1}*a = e$*.*

A group is typically denoted by the set of elements comprising it: i.e., $(\Gamma, *)$ is simply denoted Γ. The *order* of the group, denoted $o(\Gamma)$, is the number of elements in this set. If the binary operation is not explicitly stated, context will make clear which operation will be used. When we are dealing with a multiplicative group, we will use the more compact convention that $a*b = ab$. In a group Γ, an *involution* is an element $g \in \Gamma$ such that $g = g^{-1}$. Note that all groups contain at least one involution, namely the identity. Let $i(\Gamma)$ denote the number of elements in Γ that are involutions.

Definition 1.2.2 *[40] A set* $X \subseteq \Gamma$ *is a* generating set *for* Γ *if for all* $a \in \Gamma$*, there exists a finite subset* $\{x_1, x_2, ..., x_k\}$ *of* X*, such that* $a = \prod_{i=1}^{k} x_i^{m_i}$ *where* $m_i \in \mathbb{N}$*. In this case, we say that* Γ *is* generated *by* X*.*

There are many examples of groups that are described in the literature [26, 40]. We will be primarily interested in a few special cases:

(i) We say that a group Γ is *abelian* if for all $a, b \in \Gamma$, $ab = ba$ [40].

(ii) A group Γ is *cyclic* if it can be generated by a single element a [40].

(iii) If Γ and Λ are groups, the *direct product* of Γ and Λ, denoted $\Gamma \times \Lambda$, will be the set (g, ℓ) where $g \in \Gamma$ and $\ell \in \Lambda$. Multiplication will be defined as follows: For $g_1, g_2 \in \Gamma$ and $\ell_1, \ell_2 \in \Lambda$, we have $(g_1, \ell_1)(g_2, \ell_2) = (g_1 g_2, \ell_1 \ell_2)$. In this case, $(g, \ell)^{-1} = (g^{-1}, \ell^{-1})$ and the identity element of $\Gamma \times \Lambda$ is (e_Γ, e_Λ), where e_Γ and e_Λ are the identity elements of Γ and Λ respectively [40].

(iv) We denote $\Gamma \times \Gamma := \Gamma^2$. For $n \in \mathbb{Z}$ such that $n > 2$ we define Γ^n recursively by $\Gamma^n := \Gamma \times \Gamma^{n-1}$ [40].

In the literature, it is common to use C_n to denote the cyclic group of order n [26, 40]. However, this would conflict with our previous notation of C_n as the cyclic graph of order n. To this end, we establish the notation $\mathbb{Z}_n$ as the cyclic group of order n. For our purposes, $\mathbb{Z}_n$ will be assumed to be a group under addition.

There are a few notions from classical number theory that will be used occasionally throughout this document [4, 38, 40]. For completeness they are listed here.

Definition 1.2.3 *[4, 38, 40] Let a, d be integers. If d* divides *a, then there exists $c \in \mathbb{Z}$ such that $cd = a$, and this situation is denoted $d|a$. If there is no such c, we say that d does not divide a, and this situation is denoted $d \nmid a$. Let $S \subseteq \mathbb{Z}^+$. Then the greatest common divisor of S, denoted gcd(S), is the largest $d \in \mathbb{Z}^+$ such that $d|a$ for all $a \in S$.*

This notion can be used to make certain claims about cyclic groups.

Proposition 1.2.4 *[40]*

(i) $\mathbb{Z}_n \times \mathbb{Z}_m$ is cyclic if and only if gcd$(n, m) = 1$.

(ii) Any finite abelian group can be written as the direct product of cyclic groups.

Proposition 1.2.5 *[38, 40] Let* $S = \{s_1, ..., s_k\} \subseteq \mathbb{Z}^+$. $gcd(S) = 1$ *if and only if there exists* $x_1, ..., x_k \in \mathbb{Z}$ *such that:*

$$\sum_{i=1}^{k} s_i x_i = 1.$$

The form given in Proposition 1.2.5 is quite common. In fact, many of the necessary conditions that we derive will be of this form, as described in the following definition.

Definition 1.2.6 *[38] Let* $s_1, ..., s_k, x_1, ..., x_k, N \in \mathbb{Z}$. *A* Linear Diophantine Equation *is of the form:*

$$\sum_{i=1}^{k} s_i x_i = N.$$

It is known that if $gcd(s_1, ..., s_k)|N$, then such equations have an infinite number of integer solutions. However, when we restrict our attention to non-negative integer solutions, there are only finitely many solutions, perhaps none at all [38]. The assumption of non-negative integer solutions is a standard restriction for Integer Programming problems [67] as well as the problem of Frobenius [21, 25, 54, 70].

1.3 Factorizations and Matchings of Graphs

Definition 1.3.1 *A* factor *of a graph* H *is a graph* G *such that* $G \subseteq H$ *and* $V(G) = V(H)$. *A* k-factor *of a graph* H *is a* k*-regular graph* G *such that* G *is a factor of* H. *A* factorization *of a graph* H *is a set of edge disjoint factors whose edge sets partition the edge set of* H. *In particular, an* r-factorization *is a factorization into* r*-factors [3]. A* matching *of* H *is set of edges in* H *so that no two edges share a common endpoint. A* perfect matching *of* H *is a matching of* H *such that every vertex of* H *is the endpoint of exactly one of the edges. Note that a perfect matching of* H *is a 1-factor of* H *[75].*

An *isomorphic factorization* of H is a partition of the edges of H so that each part of the partition is isomorphic to a factor, G [51]. It should be noted that in order to achieve a factorization, it is admissible for G to have isolated nodes so that $V(G) = V(H)$. Note that given a factorization, we can obtain a decomposition by deleting the isolated nodes of

G. Similarly, given a G-decomposition, we can obtain a factorization by including enough isolated nodes so that $V(G) = V(H)$.

As the focus of this dissertation is graph decompositions and the resulting intersection graphs, we are rarely interested in factorizations or perfect matchings. First, a matching does not partition the edge set of H, and will not necessarily lead to a decomposition. In the case of both factorizations and matchings we have that $V(G) = V(H)$. This implies that the G-decomposition graph will necessarily be complete. However, these concepts are applicable to related concepts, particularly the idea of a vertex cover.

1.4 Combinatorial Designs

A *combinatorial design*, or "design," is a way of selecting subsets from a finite set S, such that certain conditions are met [73]. The members of S are called *treatments* and the subsets chosen are called *blocks*. It is not a coincidence that we have used the phrase "block" before in our discussions of decompositions. In fact, we have borrowed this terminology from combinatorial designs, with much the same intent. A design is a *block design* if the blocks are unordered sets of treatments. An *r-regular design* is a collection of k-sets from S such that every element of S belongs to r of the k-sets. The number of blocks is typically denoted by b [73]. Traditionally, we denote the number of treatments in S by v. A *symmetric design* is a design with $b = v$ [72].

In terms of a G-decomposition of H, we can think of the treatments in S as the nodes of the host graph H. Further, we can think of the blocks of the design as the G-blocks of the decomposition. Thus, there is a correlation between the existence of a design and the existence of a graph decomposition. However, there are restrictions to this approach. First, in traditional design theory, the treatments are interchangeable. Generally, the nodes of H are not interchangeable. Second, the literature is often concerned with block designs. In a block design, the blocks are unordered sets of treatments. This would imply that the nodes of G are also interchangeable, which is generally not the case. Also, in many results, the intersection structure of different blocks is not mentioned explicitly. As the primary concern of this dissertation is the structure of G-decomposition graphs, this is a major restriction to us.

Even with the restrictions outlined above, there are several results and definitions that are pertinent to our study. This section will be primarily concerned with such results. We also will be concerned with graphical designs, in which an extension of the complete graph is decomposed by an arbitrary graph [14, 39]. The λ-*extension* of K_n, λK_n, is a graph on n vertices where every two distinct vertices share exactly λ edges. It should be noted that because of the incredible amount of material on the subject of combinatorial designs, this section can at best offer only a brief discussion of the more important ideas.

If all v elements of S occur in a block, we say that block is *complete*. If the design contains at least one block that is not complete, then we say that the design is *incomplete*. If x and y are two different treatments in a design, then the number of blocks that contain both x and y is called the *covalency* of x and y and is denoted λ_{xy}. A design is *balanced* if λ_{xy} is constant for all choices of x and y. In such cases, we write λ as this constant and refer to it as the *index* of the balanced design. Of particular interest are *balanced incomplete block designs* or BIBD's. BIBD's are typically classified by their parameters, and in such a case we will refer to it as a (v, b, r, k, λ)-BIBD. A design is *resolvable* if the blocks can be partitioned into classes such that every treatment is contained in a class exactly once [36, 73].

Proposition 1.4.1 *[73]*

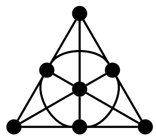
(i) In any regular design, $bk = vr$.

(ii) In any symmetric regular design, $k = r$.

(iii) In a (v, b, r, k, λ)*-BIBD, we have that* $r(k-1) = \lambda(v-1)$.

Proposition 1.4.1 allows us to reduce the number of parameters that we use in describing certain types of designs. A $t-(v, k, \lambda)$-design, or simply *t-design*, is a way of selecting blocks of size k from a v-set so that any set of t treatments appears as a subset of exactly λ blocks [59]. An important special case of the t-design is usually credited to Steiner [69], who

Figure 1.6 Projective Plane of Order 2



was unaware of the earlier work by Kirkman [55]. Let $t, k, v \in \mathbb{Z}$ be such that $2 \leq t < k < v$. A *Steiner system*, denoted $S(t, k, v)$, is a $t - (v, k, 1)$ design. In particular, a $S(2, 3, v)$ design is known as a *Steiner triple system* and is denoted $STS(v)$ [17].

A special case of the Steiner system is known as a *projective plane*. A projective plane of order p is a $S(2, p+1, p^2+p+1)$ system. A projective plane is an analog of Euclid's axioms regarding "points" and "lines" into a more general combinatorial setting. In particular, a projective plane of order p must satisfy the following three axioms:

(i) Given any $p + 1$ points, there is exactly one line between them.

(ii) Given any point, exactly $p + 1$ lines intersect at that point.

(iii) There are at least four points, no three of which are on the same line.

Figure 1.6 depicts the projective plane of order 2, also known as the Fano plane [36, 72, 73]. It easily follows from the above definitions as well as the Steiner characterization that there must be exactly $p^2 + p + 1$ points and $p^2 + p + 1$ lines in a projective plane of order p.

There are several interesting results and conjectures concerning projective planes.

Proposition 1.4.2 *[9] If n is a prime power, then there is a projective plane of order n.*

Conjecture 1.4.3 (The Prime Power Conjecture [9]) *If n is not a prime power, then there is no projective plane of order n.*

Proposition 1.4.4 (Bruck-Ryser-Chowla Theorem [9, 11]) *If* $n \equiv 1 \pmod 4$ *or* $n \equiv 2 \pmod 4$ *and* n *cannot be written as* $n = s^2 + t^2$ *where* $s, t \in \mathbb{Z}^+$*, then there is no projective plane of order* n.

Another useful class of designs is that of a configuration. A *configuration* (V_r, b_k) is an incidence structure of v points and b lines such that:

(i) Each line contains k points.

(ii) Each point lies on r lines.

(iii) Two different points are connected by at most one line.

The name *tactical configuration* is given to an incidence system in which every set is incident with exactly k elements, while every element is incident with exactly r sets [35].

Typically, citing the existence of a particular design to show the existence of a decomposition is unsatisfactory for us. Since design theorems do not often take into account the intersection structure, it is difficult for us to take advantage of these results. However, many of our arguments do have a design flavor to them. Further, we can use definitions and terminology from design theory to classify certain classes of decompositions.

Of interest however, is the idea of a graphical design. Let G be a graph. A $(\lambda K_n, G)$-*design* is a partition of the edge set of λK_n into subgraphs each of which is isomorphic to G [14, 39]. Note that the existence of a $(\lambda K_n, G)$-design implies that $G|\lambda K_n$.

Normally, we are only interested in simple graphs. Thus, the generalization to λK_n is unnecessary for us at this time. When we restrict ourselves to $\lambda = 1$, many of the existence results summarized by Heinrich [39] reduce to problems that we will solve in the next chapter. Hence, we will limit our discussion to stating the necessary conditions for the existence of such a design, and refrain from a full exposition. However, we will need to define the degree sequence of a graph. The *degree sequence* of a graph G, denoted $dseq(G)$, is an ordered list of the vertex degrees of G.

Proposition 1.4.5 *[39] If there exists a* $(\lambda K_n, G)$*-design, then:*

(i) $\lambda n(n-1) \equiv 0 \pmod{2e(G)}$.

(ii) $\lambda(n-1) \equiv 0 \pmod{d}$ *where* $d = gcd(dseq(G))$.

(iii) If the design is balanced, then $\lambda n(G)(n-1) \equiv 0 \pmod{2e(G)}$.

(iv) If the design is resolvable, then $n \equiv 0 \pmod{n(G)}$.

(v) If the design is resolvable, then there exists integers $x_1, ..., x_{n(G)}$ *such that:*

$$\sum_{i=1}^{n(G)} x_i d_i = \lambda(n-1) \quad \textit{and} \quad \sum_{i=1}^{n(G)} x_i = \frac{\lambda n(G)(n-1)}{2e(G)}$$

where $dseq(G) = [d_1, ..., d_{n(G)}]$.

1.5 Preview

The goal of this section is to outline the remainder of the dissertation as well as to highlight some of the main results.

In the next chapter, we will further examine graph decompositions. First, we will give necessary conditions for the existence of a G-decomposition of H. We will also expand Rosa's [66] notion of a valuation to that of an arbitrary cyclic group. It is well-known that Rosa's valuations will induce a decomposition of a complete graph. By extending this idea, we will obtain decompositions of more general circulant graphs. We will also discuss the structure of a G-decomposition graph. In particular, Theorem 2.4.7 will give the G-decomposition graph when the decomposition is induced by a valuation. We will also give a construction that shows that if $\Delta(K) \leq n(G)$, then K is a G-decomposition graph.

The main focus of Chapter Three will be automorphic decompositions. Using the results of Chapter Two, we will give several examples of automorphic decompositions. We will also develop a series of necessary conditions for the existence of an automorphic G-decomposition of H in Theorem 3.2.1. Some of the more important necessary conditions are:

- $e(H) = n(H)e(G)$.

- $\overline{d}(H) = 2e(G)$.
- The average number of G-blocks at a node of H is $n(G)$.
- The maximum number of G-blocks at a node of H is $\omega(H)$.
- $n(G) \leq \chi(H)$.

Using these necessary conditions, we will begin to study graphs that can host an automorphic decomposition.

Building from this, we are able to give a more complete characterization in Chapter Four. We conjecture that only even regular graphs can host an automorphic decomposition. By further refining the notion of an automorphic decomposition, we give several regularity results. Two of the most profitable assumptions are that of a simple automorphic decomposition and a fully automorphic decomposition.

In a *simple G-decomposition* of H, any two blocks may share at most one common node in H. In Theorem 4.1.1, we show that if G is d-regular and of order p and $\mathscr{D}$ is a simple automorphic G-decomposition of H, then H must be $p(p-1)$-regular. Further, we must have that G is a complete graph.

If $\mathscr{D}$ is a *fully automorphic G-decomposition* of H, we require that $I(\mathscr{D}) \cong H$ and $\chi(H) = n(G)$. In Theorem 4.2.2, we state additional necessary conditions for the existence of a fully automorphic G-decomposition of H. In particular, we show that:

- $n(G)$ blocks meet at every node of H.
- If G is not the disjoint union of P_2's, then $\delta(H) \geq n(G) + 1$.

By applying these necessary conditions, we will be able to show the following regularity results in Theorem 4.2.4:

- If $\chi(H) \leq 3$, then H is $2e(G)$-regular.
- If $\chi(H) = n(G)$ and G is d-regular, then H is $n(G)d$-regular.

- If $\chi(H) = n(G)$, G is not a disjoint union of P_2's, and the smallest elements of $dseq(G)$ are 1 and a, where $a \geq 2e(G) - n(G) + 1$, then H must be $2e(G)$-regular.

- If $G \cong P_4$ and $\chi(H) = 4$, then H is 6-regular.

In Chapter Five, we examine the admissible prototypes for an automorphic G-decomposition. If a graph is an admissible prototype for an automorphic decomposition, we say that the graph is an *automorphic divisor*. We conjecture that every graph is an automorphic divisor. By further generalizing valuations to arbitrary groups, we are able to take direct products of the valuations. This allows us to extend the class of graphs known to be automorphic divisors. Also of interest is which graphs are automorphic divisors for an infinite number of non-isomorphic hosts. Finally, we investigate what graphs are fully automorphic divisors.

In Chapter Six, we look at various ways of extending automorphic decompositions. In particular, we look at the possibilities and consequences if we allow G and H to be multigraphs. We will also consider the possibility of decomposition with respect to a family of prototypes.

In the final chapter, we will summarize some of our more important results. We will also restate several open problems.

CHAPTER 2

GRAPH DECOMPOSITIONS

In the previous chapter, we alluded to graph decompositions. Given a graph H (called the *host*) and a graph G (called the *prototype*), a *G-decomposition* of H, denoted $\mathscr{D}$, is a partition of the edge set of H such that the subgraph induced by each part of the partition (called *blocks*) is isomorphic to G. Except in certain cases, the problem of whether there exists a G-decomposition of H is difficult and unsolved [9, 75].

We are especially interested in the notion of an intersection graph generated by the decomposition $\mathscr{D}$. The *intersection graph* generated by a decomposition $\mathscr{D}$, denoted $I(\mathscr{D})$, has a vertex for each block in the decomposition and two vertices are adjacent in $I(\mathscr{D})$ if and only if the corresponding blocks share a common node in H. We will often refer to an intersection graph generated by a G-decomposition as a *G-decomposition graph*.

There has been a tremendous amount of research on the chromatic number of G-decomposition graphs [7, 15, 16, 18, 19, 23, 24, 22, 28, 34, 45, 46, 47, 62, 63]. Specifically, the problem of the chromatic spectrum considers all possible values of $\chi(I(\mathscr{D}))$ where $\mathscr{D}$ is a G-decomposition of H.

In this dissertation, we are concerned with a finer problem than that of the chromatic spectrum. In particular, we are interested in whether the G-decomposition graph can ever be isomorphic to the host. As the problem of graph isomorphism is a notoriously difficult one [5], to say nothing of the inherent difficulties of determining the existence of such a decomposition, we expect this to be a difficult problem. The goal of this chapter is to develop the mathematical machinery necessary to attack this problem. We will also exhibit some properties about decompositions and their intersection graphs.

2.1 Labellings and Valuations on Graphs

We will often be labelling a graph with the elements of a group, usually a cyclic one, or at least the direct product of cyclic groups.

Let $\mathbb{Z}_n$ denote the integers modulo n. We can think of the "positive" elements of this group as $\mathbb{Z}_n^+ := \{x \in \mathbb{Z}_n : 0 < x < n/2\}$. In many of our arguments and constructions, the involution $n/2$ creates problems, and for this reason it is excluded from $\mathbb{Z}_n^+$. However, on occasion, it is useful to include $n/2$. In which case, we will define the set $\mathbb{Z}_n^* := \{x \in \mathbb{Z}_n : 0 < x \leq n/2\}$. It is useful to introduce a *modular absolute value* on $\mathbb{Z}_n$ as follows:

$$|x|_n = \begin{cases} x & \text{if } 0 \leq x \leq n/2 \\ n - x & \text{if } n/2 < x < n. \end{cases}$$

Definition 2.1.1 *Let $G = (V, E)$ be a graph. A $\mathbb{Z}_n$-*labelling *of G is an injective map $f : V \to \mathbb{Z}_n$. Any $\mathbb{Z}_n$-labelling f of G induces an edge labelling f^* on E by $f^*(xy) = |f(x) - f(y)|_n$ for all edges $xy \in E$. We denote the set of edge labels induced by f by:*

$$f^*(E) = \{|f(x) - f(y)|_n : xy \in E\}.$$

Similarly, this concept can be extended to all pairs of vertices with:

$$f^*(G) = \{|f(x) - f(y)|_n : x, y \in V, x \neq y\}.$$

Note that $f^*(E) \subseteq f^*(G)$ for all f and $G = (V, E)$.

Definition 2.1.2 *A $\mathbb{Z}_n$-labelling f of G is a $\mathbb{Z}_n$-*valuation *if and only if the induced edge labelling f^* is injective and, if n is even, the involution $n/2$ does not appear as an edge label.*

Remark 2.1.3 *It is possible to extend the idea of a valuation to other groups, even non-abelian ones. This idea will further be explored in a later chapter.*

It is customary to regard $\mathbb{Z}_n$ formally as congruence classes of $\mathbb{Z}$ modulo the ideal generated by n. Informally, $\mathbb{Z}_n$ is just thought of as the set of integers with certain identifications. The convention here to identify $\mathbb{Z}_n$ formally with a particular choice of canonical representatives of the congruence classes, namely, $\{0, 1, 2, 3, \ldots, n-1\}$ has certain advantages. One of these is that $\mathbb{Z}_n$ is actually a subset of $\mathbb{Z}_m$ if $n \leq m$. It does not inherit the algebraic structure, of course. But this fact makes it possible to say that a $\mathbb{Z}_n$-labelling of a graph G is also a $\mathbb{Z}_m$-labelling of G for all $m \geq n$. A $\mathbb{Z}_n$-valuation of a graph G need not be a $\mathbb{Z}_m$-valuation of G even when $m \geq n$ because of the change in algebraic structure. But, under certain circumstances, a $\mathbb{Z}_n$-valuation of G can be a $\mathbb{Z}_m$-valuation of G. These considerations lead to several new types of valuations (persistent, positive, closed, replete, and complete) as well as unify several classical forms of valuations ($\alpha-, \beta-$, $\rho-$, and $\sigma-$) introduced by Rosa [6, 66].

Definition 2.1.4 *[6, 66] Let f be a $\mathbb{Z}_n$-valuation of a graph $G = (V, E)$ with q edges.*

(i) f *is* persistent *if and only if* f *is a* $\mathbb{Z}_m$*-valuation of* G *for all* $m \geq n$.

(ii) f *is* positive *if and only if* $f : V \to \mathbb{Z}_n^+ \cup \{0\}$.

(iii) f *is* closed *if and only if* $f^*(G) \subseteq f^*(E)$.

(iv) f *is* replete *if and only if* $f^*(G) \supseteq \mathbb{Z}_n^+$.

(v) f *is* complete *if and only if* $f^*(E) = \mathbb{Z}_n^+$.

(vi) f *is a* ρ-valuation *if and only if* $n = 2q + 1$.

(vii) f *is a* β-valuation *if and only if* f *is a positive* ρ*-valuation.*

Of the valuations listed above, ρ-valuations and β-valuations are the most well studied [31]. Note that Rosa's β-valuations [66] were popularized by Golomb [32] who called such labellings *graceful*. As this is the popular term, we will refer to β-valuations as graceful labellings in the future. There are many other types of labellings/valuations on graphs that have been studied. Some of these include α-valuations, σ-valuations [66], almost graceful labellings [64], and nearly graceful labellings [30]. Each of these labellings imply a ρ-valuation. However, they do not fit readily into the notation outlined above. Further, the definitions presented above can be readily generalized to arbitrary groups, while others are more dependent on the integers. The interested reader is referred to the original articles as well as Gallian [31] for the definition of these labellings as well as examples.

Proposition 2.1.5 *Let G be a finite simple graph of size q.*

(i) If f is a ρ-valuation on G, then f is a closed $\mathbb{Z}_{2q+1}$-valuation on G.

(ii) If f is a graceful labelling on G, then f is a closed $\mathbb{Z}_n$-valuation of G for $n \geq 2q+1$.

Proof.

(i) By definition of ρ-valuation, each of the edges of G is labelled with a distinct element of $\mathbb{Z}^+_{2q+1}$. Thus, if f is a ρ-valuation of G, it follows that $f^*(E(G)) = \mathbb{Z}^+_{2q+1}$. Hence by definition, if $u, v \in V(G)$ then $|f(u) - f(v)|_{2q+1} \in \mathbb{Z}^+_{2q+1}$. Ergo, $f^*(G) \subseteq f^*(E(G))$. As such, f is closed by definition.

(ii) Since every graceful labelling is a ρ-valuation [66], it follows from (i) that if f is a graceful labelling, then f is a closed $\mathbb{Z}_{2q+1}$-valuation. Further, since each element of $\mathbb{Z}^+_{2q+1}$ is represented as an edge label exactly once and $f(V) \subseteq \mathbb{Z}^+_{2q+1} \subseteq \mathbb{Z}^+_n$ for all $n \geq 2q+1$, it follows that $f^*(G) \subseteq \mathbb{Z}^+_{2q+1} \subseteq \mathbb{Z}^+_n$ for all $n \geq 2q+1$. Thus, f is a closed $\mathbb{Z}_n$-valuation of G for all $n \geq 2q+1$.

■

Proposition 2.1.6 *The following graphs are known to have a graceful labelling:*

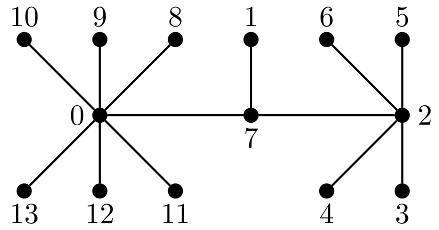
(i) Caterpillars [66].

(ii) Trees with at most four endpoints [43, 53, 66, 77].

(iii) Trees in which, at any two vertices u and v, there is a P_k with u and v as its endpoints, where $k \leq 5$ [42, 77].

(iv) Trees of order at most 27 [2].

Figure 2.1 A Gracefully Labelled Caterpillar - $C(6,1,4)$



(v) Complete bipartite graphs [32, 66].

(vi) Cycles of length n where $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ [56].

(vii) Gear graphs, the Petersen graph, and polyhedral graphs [74].

(viii) $C_n \vee K_1$ (i.e., Wheel Graphs) [29].

(ix) The n-dimensional hypercube, Q_n [57, 60].

(x) The graph obtained by subdividing each edge of a graceful tree [12].

It has been conjectured by Rosa [66] that all trees have a graceful labelling. This is known as the *Graceful Tree Conjecture*. Examples of graceful labellings on trees are given in Figure 2.1 and Figure 2.2.

Note that a graceful labelling is never unique. If f is a graceful labelling on a graph with q edges, a new graceful labelling g may be obtained by taking $g(v) = q - f(v)$. This is a common technique, often referred to as the *reversal* of a graceful labelling. However, there are often several graceful labellings for a given graph, other than the reversal. Figure 2.3 gives several graceful labellings for the graph shown in the upper left corner of Figure 2.2.

The following is a short list of graphs (not included above) that have a ρ-valuation.

Proposition 2.1.7 *The following graphs are known to have a ρ-valuation.*

(i) The graphs listed in Proposition 2.1.6 [66].

Figure 2.2 Graceful Labellings For Non-Caterpillar Trees With $n(G) \leq 9$

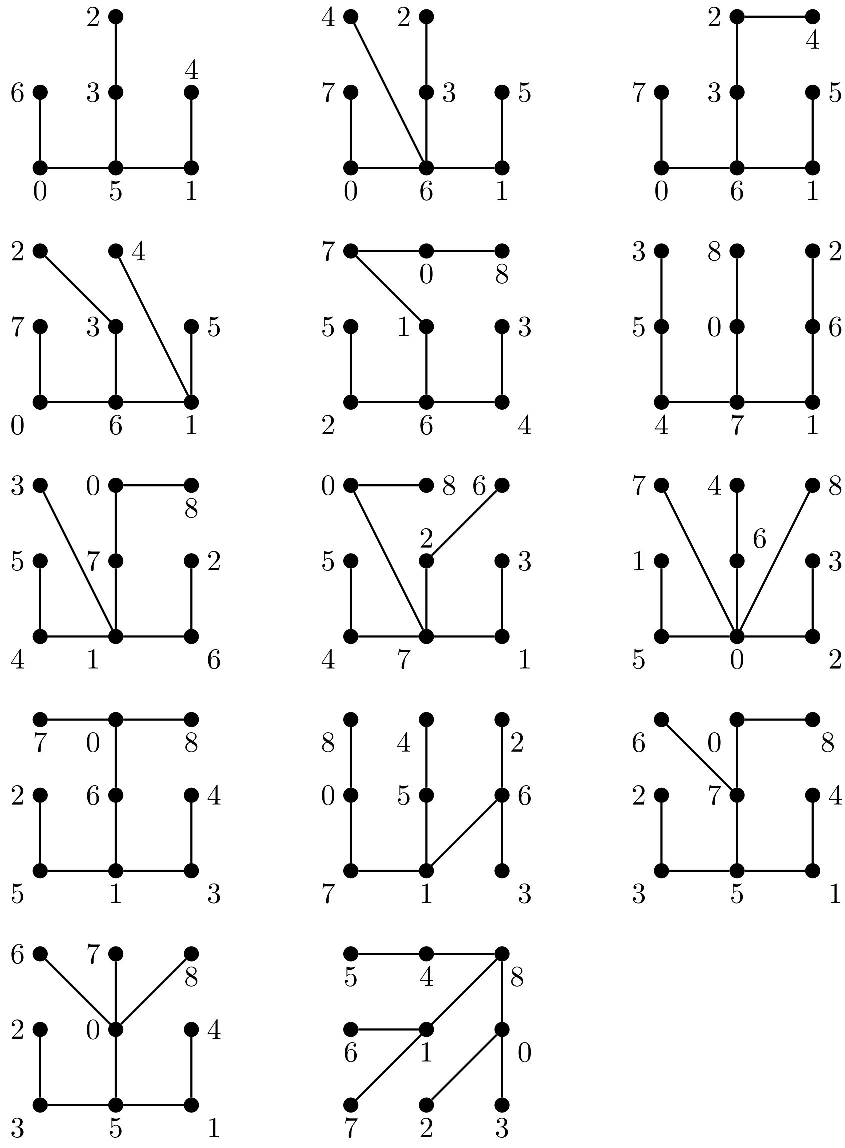


Figure 2.3 Graceful Labellings of a Non-Caterpillar Tree With $n(G) = 7$

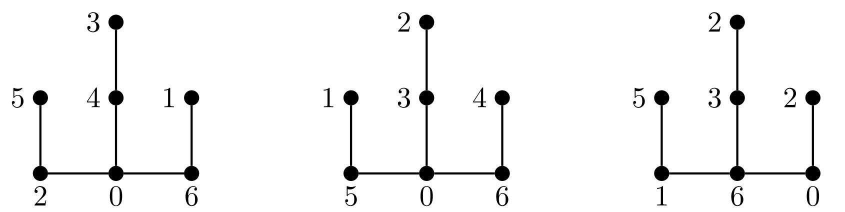
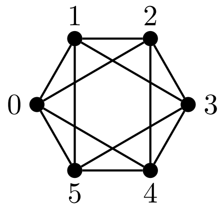


Figure 2.4 Circulant - $C_6(1, 2)$



(ii) *Cycles [68].*

(iii) *Graphs G such that G_A has a graceful labelling, where $A = \{v \in V(G) : deg_G(v) \geq 2\}$ [66].*

(iv) *All graphs with at most eleven edges [13].*

2.2 Circulants

Throughout this document, we will frequently refer to a special class of graphs known as circulants. Because of the importance of these graphs to our results and the fact that these graphs are not mentioned in the standard texts, we will devote more time to the properties of such graphs.

Definition 2.2.1 *[41, 62] Let $n \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}_n^*$ be given. The* circulant graph $C_n(S)$ *is the undirected graph with vertex set $V = \mathbb{Z}_n$ and edge set*

$$E = \{xy : x, y \in \mathbb{Z}_n \text{ and } |x - y|_n \in S\}.$$

It is customary to refer to the elements of S as *lengths*. We say that $uv \in E(C_n(S))$ is an edge of length a if $|u - v|_n = a \in S$.

The following properties of the circulant were presented in Heuberger [41] without proof. For completeness, we have included the proofs.

Proposition 2.2.2 *Let $n \in \mathbb{Z}$, $n \geq 3$ and $S = \{a_1, ..., a_m\} \subseteq \mathbb{Z}_n^*$ be given.*

(i) $C_n(S)$ *is connected if and only if* $gcd(a_1, ..., a_m, n) = 1$.

(ii) $C_n(S)$ *with* $gcd(a_1, ..., a_m, n) = d$ *is isomorphic to* d *disjoint copies of* $C_{n/d}(a_1/d, ..., a_m/d)$.

(iii) If $gcd(a_j, n) = 1$ *for some* $j = 1, ..., n$ *and* $a_j^{-1} := b \pmod{n}$ *then:*

$$C_n(a_1, .., a_m) \cong C_n(a_1 b, ..., a_m b).$$

(iv) $C_n(S)$ *is d-regular where:*

$$d = \begin{cases} 2m & \text{if } S \subseteq \mathbb{Z}_n^+ \\ 2m-1 & \text{otherwise.} \end{cases}$$

(v) $C_n(a) \cong C_n$ *if and only if* $gcd(a, n) = 1$.

(vi) $C_n(1, 2, ..., \lfloor n/2 \rfloor) \cong K_n$.

(vii) $\overline{C_n(S)} \cong C_n(\mathbb{Z}_n^* - S)$.

Proof.

(i) Note that $C_n(S)$ is connected if and only if there is a walk between any two distinct vertices. This happens if and only if there is a sequence of edge lengths that sum to one modulo n. This happens if and only if there are integers $x_1, ..., x_m$ such that:

$$x_1 a_1 + \cdots + x_m a_m \equiv 1 \pmod{n}.$$

This is equivalent to the existence of an integer p such that:

$$x_1 a_1 + \cdots + x_m a_m = 1 + pn$$

$$\Leftrightarrow x_1 a_1 + \cdots + x_m a_m - pn = 1.$$

This is equivalent to $gcd(a_1, ..., a_m, n) = 1$ by Proposition 1.2.5.

(ii) Suppose that $gcd(a_1, ..., a_m, n) = d$. This implies that there exists integers $x_1, ..., x_m, p$ such that:

$$x_1a_1 + \cdots + x_ma_m + pn = d$$

$$\Leftrightarrow x_1a_1 + \cdots + x_ma_m = d - pn$$

$$\Leftrightarrow x_1a_1 + \cdots + x_ma_m \equiv d \pmod{n}.$$

Thus, any sequence of edge lengths will result in a walk whose length is a multiple of $d \pmod{n}$. Ergo, if $v \in V(C_n(S))$, and C_v is the connected component containing v, then the elements of C_v are of the form $v + kd$ where $k \in \mathbb{Z}$. Since $d|n$, it follows that there are d such connected components, each containing $\frac{n}{d}$ vertices. Note that the above equation is equivalent to:

$$x_1\frac{a_1}{d} + \cdots + x_m\frac{a_m}{d} \equiv 1 \pmod{n}.$$

It remains to be shown that C_v is isomorphic to $C_{n/d}(a_1/d, ..., a_m/d)$. Let $f : V(C_v) \to V(C_{n/d}(a_1/d, ..., a_m/d))$ be such that $f(x) = \frac{x}{d} \pmod{\frac{n}{d}}$ for all $x \in V(C_v)$. Note that if $uv \in E(C_n(S))$, then $|u - v|_n = a_j \in S$. So,

$$f(uv) = |f(u) - f(v)|_{n/d} = |\frac{u}{d} - \frac{v}{d}|_{n/d} = \frac{a_j}{d}.$$

Thus, $f(uv) \in C_{n/d}(a_1/d, ..., a_m/d)$. Hence, $C_v \cong C_{n/d}(a_1/d, ..., a_m/d)$.

(iii) It suffices to exhibit the required bijection. Let $f(v) = a_j^{-1}v \pmod{n}$ for $v \in C_n(S)$. Since $gcd(a_j, n) = 1$, it follows that this is a bijective function. Suppose that $uv \in E(C_n(S))$. This is equivalent to:

$$|u - v|_n \in \{a_1, ..., a_m\}.$$

Note that:

$$f(uv) = |f(u) - f(v)|_n = |a_j^{-1}u - a_j^{-1}v|_n$$

$$= a_j^{-1}|u - v|_n \in \{a_1a_j^{-1}, ..., a_ma_j^{-1}\}.$$

It follows that the two graphs are isomorphic.

(iv) Let $v \in C_n(S)$. Note that v is adjacent to vertices of the form $v \pm a_j \pmod{n}$. Further note that these vertices are distinct, provided that $a_j \neq \frac{n}{2} \pmod{n}$. Since there are m distinct a_j, it follows that v is adjacent to $2m$ other vertices provided that none of these differences is $\frac{n}{2}$. If one of the differences is $\frac{n}{2}$, then v is adjacent to $2m-1$ other vertices. Since v was chosen arbitrarily, it follows that $C_n(S)$ is d-regular where:

$$d = \begin{cases} 2m & \text{if } S \subseteq \mathbb{Z}_n^+ \\ 2m-1 & \text{otherwise.} \end{cases}$$

(v) If $gcd(a,n) = d > 1$, then by (ii), $C_n(a)$ is isomorphic to d disjoint copies of $C_{n/d}(a/d)$ and as such cannot be isomorphic to a connected cycle.

If $gcd(a,n) = 1$, then $C_n(a) \cong C_n(1)$ by (iii). Since $n \geq 3$, it follows that $1 \neq \frac{n}{2}$. Thus by (iv), it follows that $C_n(1)$ is a 2-regular connected graph. Since the only 2-regular connected graphs are cycles by Proposition 1.1.3, it follows that $C_n(1) \cong C_n$.

(vi) If n is even, then $C_n(1, ..., \frac{n}{2})$ is $2(\frac{n}{2}) - 1 = n-1$ regular by (iv). Similarly if n is odd, then $C_n(1, ..., \frac{n-1}{2})$ is $2(\frac{n-1}{2}) = n-1$ regular by (iv). Since the only simple graphs of order n that are $n-1$ regular are complete, it follows from either case that $C_n(1, 2, ..., \lfloor n/2 \rfloor) \cong K_n$.

(vii) Note:

$$uv \in E(C_n(S)) \Leftrightarrow |u-v|_n \in S$$

$$\Leftrightarrow |u-v|_n \notin \mathbb{Z}^* - S$$

$$\Leftrightarrow uv \notin E(C_n(\mathbb{Z}^* - S)).$$

Thus, it follows that $\overline{C_n(S)} \cong C_n(\mathbb{Z}_n^* - S)$. ∎

For circulants that are not isomorphic to either cycles or complete graphs, very few general results about the chromatic number are known. We present the results that are known as of the time of this writing.

Using the Hermite Normal Form of a matrix as described in Schrijver [67] as well as Brooks' Theorem [10], Heuberger [41] was able to prove the following result about the colorability of circulants defined by a set of two distinct differences.

Proposition 2.2.3 *[41] Let $G = C_n(a, b)$ be a connected circulant with $|a|_n \neq |b|_n$. Then:*

(i) $\chi(G) = 2$ *if and only if* a *and* b *are odd and* n *is even.*

(ii) $\chi(G) = 4$ *if* $3 \nmid n$, $n \neq 5$ *and* $b \equiv \pm 2a \pmod{n}$ *or* $a \equiv \pm 2b \pmod{n}$.

(iii) $\chi(G) = 4$ *if* $n = 13$ *and* $b \equiv \pm 5a \pmod{13}$ *or* $a \equiv \pm 5b \pmod{13}$.

(iv) $\chi(G) = 5$ *if* $n = 5$.

(v) $\chi(G) = 3$ *in all other cases.*

Building on the work of Albertson and Hutchinson [1], Yeh and Zhu [76], as well as Collins and Hutchinson [20], the following result was proven by Meszka, Nedela, and Rosa [62] in order to color Steiner triple systems.

Proposition 2.2.4 *[62] Let $G = C_n(a, b, a + b)$ be a connected 6-regular circulant, where $n \geq 7$ and $a, b,$ and $a + b$ are pairwise distinct positive integers. Then:*

(i) $\chi(G) = 7$ *if and only if* $n = 7$.

(ii) $\chi(G) = 6$ *if and only if* $G \cong C_{11}(1, 2, 3)$.

(iii) $\chi(G) = 5$ *if and only if* $G \cong C_n(1, 2, 3)$ *and* $n \neq 7, 11$ *is not divisible by 4 or G is isomorphic to one of the following circulants:* $C_{13}(1, 3, 4)$, $C_{17}(1, 3, 4)$, $C_{18}(1, 3, 4)$, $C_{19}(1, 7, 8)$, $C_{25}(1, 3, 4)$, $C_{26}(1, 7, 8)$, $C_{33}(1, 6, 7)$, $C_{37}(1, 10, 11)$.

(iv) $\chi(G) = 3$ *if and only if* $3|n$ *and* $3 \nmid a, b, a + b$.

(v) $\chi(G) = 4$ *in all other cases.*

While these theorems give the exact chromatic number of circulants with a small difference set, we will later need a bound on the chromatic number of circulants with a larger number of differences. Based on the proofs of the above results, we give a general upper bound in the next proposition.

Proposition 2.2.5 *Let $G = C_{kn}(S)$ be a circulant such that for all $a \in S$, $k \nmid a$ and n is sufficiently large. Then we have $\chi(G) \leq k$.*

Proof.

Let $V(G) = \{0, 1, ..., kn - 1\}$. We claim that the required color classes are the congruence classes modulo k. Since none of the differences are divisible by k, it follows that no two elements in a congruence class share a common edge. Thus, we need at most k colors. ■

By observing how cliques are formed within a circulant, a lower bound for the chromatic number can be achieved.

Proposition 2.2.6 *Let $S \subseteq \mathbb{Z}_n^+$ be given. $\omega(C_n(S)) \geq k + 1$ if and only if there exists $\{a_1, ..., a_k\} \subseteq S$ such that $|a_i - a_j|_n \in S$ for all $1 \leq i < j \leq k$.*

Proof.

Let $H = C_n(S)$ and suppose that $\omega(H) \geq k + 1$. This implies that H contains K_{k+1} as a vertex induced subgraph. Let $A \subseteq V(H)$ be such that $H_A \cong K_{k+1}$. By the cyclic nature of the circulant, we may assume that $0 \in A$. Let $A = \{0, v_1, ..., v_k\}$. Since 0 is adjacent to all of the v_i, it follows that $|v_i|_n \in S$ for all i. We claim that $\{|v_1|_n, ..., |v_k|_n\}$ is the required set of differences. Since $\{0, v_1, ..., v_k\}$ are vertices of a circulant and mutually adjacent, it follows that $|v_i - v_j|_n \in S$ for all i and j.

Conversely, assume that $\{a_1, ..., a_k\} \subseteq S$ is such that $|a_i - a_j|_n \in S$ for all $1 \leq i < j \leq k$. In order to show that $C_n(S)$ contains a clique of order $k + 1$, we need to find $k + 1$ distinct mutually adjacent vertices. We claim that $\{0, a_1, ..., a_k\}$ is the required set of vertices. Clearly, 0 is adjacent to every vertex in the set as $a_i \in S$ for all i. Further, $|a_i - a_j|_n \in S$ for all $1 \leq i < j \leq k$ by definition of S. Thus, all of the vertices in $\{0, a_1, ..., a_k\}$ are mutually adjacent. As such, we have constructed the required clique. ■

2.3 Decompositions of Graphs

Definition 2.3.1 *A* decomposition $\mathscr{D}$ *of a graph* H *is a partition of the edge set* $E(H)$ *of* H*. The graph* H *is called the* host graph *for the decomposition. For each part* $\mathcal{P}$ *of the partition, the subgraph of* H *induced by* $\mathcal{P}$ *is called a* block *of the partition. We will be concerned with the case that all blocks are isomorphic to a single* block prototype G*. In this case, we say that* $\mathscr{D}$ *is a* G*-decomposition of* H *and that* G *is a* divisor *of* H*. This situation is denoted by* $G|H$*.*

It has been conjectured by Ringel that any tree of size q will decompose K_{2q+1} [65]. *Ringel's Conjecture*, as it came to be known, was the primary motivation for Rosa to introduce his valuations, outlined previously [66].

Note that the definition of an edge decomposition can readily be extended to families of graphs, rather than a single prototype. This situation will be discussed in the last chapter.

We now introduce several special types of decompositions.

Definition 2.3.2 *[9] A* G*-decomposition of* H*,* $\mathscr{D}$*, is said to be* balanced *(or more specifically* r*-balanced) if every node of* H *is in* r G*-blocks. If* H *admits a balanced* G*-decomposition, we write* $G\|H$*.*

Proposition 2.3.3 *[9] Let* G *and* H *both be regular. Then every* G*-decomposition of* H *is balanced.*

Definition 2.3.4 *A* G*-decomposition of* H*,* $\mathscr{D}$*, is said to be* simple *if given any two* G*-blocks* $B_1, B_2 \in \mathscr{D}$*,* B_1 *and* B_2 *share at most one common node in* H*.*

Simple decompositions will prove useful in certain counting arguments. This concept can be generalized by defining λ-uniform decompositions and λ-bounded decompositions. A G-decomposition $\mathscr{D}$ is λ*-uniform* if any two intersecting G-blocks share exactly λ common nodes in H. A G-decomposition $\mathscr{D}$ is λ*-bounded* if any two G-blocks share at most λ common nodes in H.

Proposition 2.3.5 *If $\mathscr{D}$ is a simple decomposition of H with respect to a simple graph G, then it follows that H is simple.*

Proof.

If $\mathscr{D}$ is a simple G-decomposition of H, then any two G-blocks share at most one common node. Take $B \in \mathscr{D}$ and edge $uv \in E(B)$. Since $\mathscr{D}$ is simple, it follows that the edge uv is unique to B. Since G is a simple graph, uv is not a multiple edge in H. Since uv was chosen arbitrarily, it follows that H is a simple graph. ■

Proposition 2.3.6 *Let $\mathscr{D}$ be a K_p-decomposition of H. $\mathscr{D}$ is a simple decomposition if and only if H is a simple graph.*

Proof.

Let $\mathscr{D}$ be a simple K_p-decomposition of H. Since K_p is simple it follows from Proposition 2.3.5 that $\mathscr{D}$ is simple. Let H be simple and take $\mathscr{D}$ to be a K_p-decomposition of H. Assume to the contrary that there are two K_p blocks, A and B, that share at least two common nodes, say u and v. Since A and B are complete, it follows that $uv \in E(A)$ and $uv \in E(B)$. Since $\mathscr{D}$ is an edge partition, it follows that uv is a multiple edge in H. This contradicts the assumption that H is simple. ■

Theorem 2.3.7 (Necessary Conditions [9]) *Let G and H be non-empty finite graphs with $V(G) = \{u_1, ..., u_{n(G)}\}$ and $V(H) = \{v_1, ..., v_{n(H)}\}$. If there exists a G-decomposition of H into b subgraphs, then we require that:*

(i) $n(G) \leq n(H)$ *or* $e(H) = 0$.

(ii) $e(G)b = e(H)$.

(iii) For each $i \in \{1, ..., n(H)\}$ *there exist non-negative integers* $x_{i,1}, ..., x_{i,n(G)}$ *such that:*

$$\sum_{j=1}^{n(G)} x_{i,j} deg_G(u_j) = deg_H(v_i).$$

(iv) $gcd(dseq(G))|gcd(dseq(H))$.

(v) $\delta(G) \leq \delta(H)$.

(vi) If in addition this decomposition is r-balanced, we require that $n(G)b = n(H)r$.

Other quantities of interest for a G-decomposition of H are the maximum number of blocks at a node of H and the average number of blocks over all nodes of H. While the minimum number of blocks may be interesting to some, it is difficult to list non-trivial bounds for this quantity. If $\mathscr{D}$ is a G-decomposition of H, then $M(\mathscr{D})$ will denote the maximum number of G-blocks at a node of H. Further, $b(\mathscr{D})$ will denote the average number of G-blocks over all nodes of H.

Proposition 2.3.8 *Let $\mathscr{D}$ be a G-decomposition of H.*

(i) $b(\mathscr{D}) = \frac{\bar{d}(H)}{\bar{d}(G)}$.

(ii) $M(\mathscr{D}) \leq \frac{\Delta(H)}{\delta(G)}$.

Proof.

(i) Note that by Theorem 2.3.7, the total number of G-blocks in a G-decomposition of H is given by $\frac{e(H)}{e(G)}$. Since each G-block is incident with $n(G)$ distinct nodes of H, the average number of blocks over all nodes of H is given by:

$$b(\mathscr{D}) = \frac{e(H)n(G)}{e(G)n(H)} = \frac{\bar{d}(H)n(H)n(G)/2}{\bar{d}(G)n(G)n(H)/2}$$
$$= \frac{\bar{d}(H)}{\bar{d}(G)}.$$

(ii) Let $V(G) = \{u_1, ..., u_{n(G)}\}$ and take $v \in V(H)$ to be a node where the maximum number of G-blocks meet. By Theorem 2.3.7, there exist non-negative integers, $x_1, ..., x_{n(G)}$ such that:

$$\sum_{i=1}^{n(G)} x_i deg_G(u_i) = deg_H(v).$$

Note that the sum $x_1 + \cdots + x_{n(G)}$ gives the number of G-blocks around v. Thus, $M(\mathscr{D}) = x_1 + \cdots + x_{n(G)}$. Applying this yields:

$$M(\mathscr{D})\delta(G) = x_1\delta(G) + \cdots + x_{n(G)}\delta(G)$$

$$\leq \sum_{i=1}^{n(G)} x_i deg_G(u_i) = deg_H(v) \leq \Delta(H)$$

$$\Rightarrow M(\mathscr{D}) \leq \frac{\Delta(H)}{\delta(G)}.$$

■

While the necessary conditions given above are relatively easy to prove using simple counting arguments, sufficient conditions are usually more difficult. Thus, it is useful to give an algebraic construction of the decomposition. Given that H satisfies certain properties, we may use the following definition to construct a G-decomposition of H.

Let $H = (W, F)$ be a graph and suppose f is an automorphism of H. Then f can be regarded as both a bijection of the nodes W and edges F of H. The powers f^n of f are defined in the standard way: f^0 denotes the identity map and $f^i = f \circ f^{i-1}$. An automorphism f is *cyclic* if and only if f is a cyclic permutation on the nodes W of H [6, 9].

Suppose $G = (V, E)$ is a (not necessarily induced) subgraph of H. Then the *image* $f[G]$ of G under f has node set

$$V(f[G]) := \{f(v) : v \in V = V(G)\}$$

and edge set

$$E(f[G]) := \{f(e) : e \in E = E(G)\}.$$

A decomposition $\mathscr{D}$ of a graph H of order n into isomorphic copies of G is *metacyclic* [9] if and only if there exists an automorphism f of H and a block $A_0 \in \mathscr{D}$ such that every other block $B \in \mathscr{D}$ appears as an image of A_0 under f. That is, $A_j = f^j[A_0]$ for $0 \leq j < n$ is a complete list of the blocks of $\mathscr{D}$. If the automorphism f is itself a cyclic automorphism of H, then the decomposition is *cyclic* [6, 9].

Example 2.3.9 *Consider the following P_3-decomposition of $C_6(1,2)$. Beginning with the P_3-block with nodes labelled* 013 *respectively, we add one to each of the components of this block, doing all of our computations modulo 6. Thus, we obtain the following blocks* 013, 124, 235, 340, 451, *and* 502. *Note that this is a valid decomposition of $C_6(1,2)$.*

Suppose that we have identified the vertices of G as $v_1, ..., v_{n(G)}$. One compact way of listing the labels on the graph is to represent the labels as $f_1, ..., f_{n(G)}$, where f_i is the label on v_i. When applicable, we will use the structure of G to determine the representation, i.e., the vertices of paths and cycles will be listed in the order they appear on the graph, the center of a star will be listed first, etc. This is applicable when discussing labellings and valuations of G, as we will often list this array with the appropriate labels in place. In this case, we will refer to positions within the array.

A generalization of a theorem proved by Rosa [66] follows:

Theorem 2.3.10 *[6] A cyclic G-decomposition of $H = C_n(S)$ exists if and only if there is a $\mathbb{Z}_n$-valuation of G such that $f^*(E(G)) = S$.*

Proof.

Let $\mathscr{D}$ be a cyclic G-decomposition of H. Let G_1 be a G-block in the decomposition. We must show that the edges of G_1 have distinct lengths in H. Suppose that G_1 has two edges of the same length. Let $(x, x + a_i), (y, y + a_i) \in E(G_1)$ where $a_i \in S$ and $x \neq y$. Without loss of generality, we may assume that $y > x$. By definition of cyclic decomposition, there is a cyclic permutation f such that if G_2 is any other G-block in the decomposition, there exists a $j \in \mathbb{N}$ such that $G_2 = f^j[G_1]$. In particular, this holds if $G_2 = f^{y-x}[G_1]$. Since $(x, x+a_i) \in E(G_1)$, it follows that $f^{y-x}(x, x+a_i) \in G_2$. But, by definition, $f^{y-x}(x, x+a_i) = (y, y + a_i) \in G_1$. Thus, the edge $(y, y + a_i)$ has been partitioned twice, contrary to $\mathscr{D}$ being an edge decomposition. Thus, the edge lengths on G_1 must be unique, and as such we have a $\mathbb{Z}_n$-valuation of G.

Let f be a $\mathbb{Z}_n$-valuation of G. For $v_i \in V(G)$, let $f(v_i) = f_i$. Denote the nodes of H in such a way that $v_i = v_{f_i}$. In other words, each node of G is identified with a unique node of H. For $v_iv_j \in E(G)$, let $d_{ij} = f^*(v_iv_j)$ be the induced edge label of v_iv_j. Since each node of G is identified with a unique node of H, it follows that d_{ij} is an edge length in H. This implies that the edges of G have in H mutually distinct lengths. This implies the existence of a cyclic decomposition. ■

It has been conjectured by Kotzig and Rosa, that if T is a tree of size q then K_{2q+1} admits a cyclic T-decomposition. Note that this would be implied by the Graceful Tree Conjecture and would in turn imply Ringel's Conjecture [66].

Theorem 2.3.11 *[6] K_p has a $\mathbb{Z}_n$-valuation for all $n \geq 2^p - 1$.*

Proof.

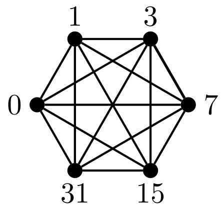
Let $V(K_p) = \{v_0, ..., v_{p-1}\}$. Let $b \in \mathbb{N}$ and $b \geq 2$. Define $f(v_i) = b^i - 1$ and $n \geq 2b^{p-1} - 1$. We claim that f is the required $\mathbb{Z}_n$-valuation. Clearly, all the vertex labels are distinct modulo $n \geq 2b^{p-1} - 1$. For distinct edges, $v_iv_j, v_kv_\ell \in E(K_p)$, where $i > j$ and $k > \ell$, the respective edge labels are:

$$f^*(v_iv_j) = b^j(b^{i-j} - 1) \quad \text{and} \quad f^*(v_kv_\ell) = b^\ell(b^{k-\ell} - 1).$$

Since the largest edge label is $b^{p-1} - 1$ and the group size is more than twice that, it follows that we need not consider the possibility that one of these edge labels is the negative of the other in the given modulus. Further, none of the edges are labelled with an involution. So we need only consider the possibility that $b^j(b^{i-j} - 1) = b^\ell(b^{k-\ell} - 1)$. Without loss of generality, assume that $j \geq \ell$. So we have $b^{j-\ell}(b^{i-j} - 1) = b^{k-\ell} - 1$. If $j > \ell$, then the left hand side is congruent to 0 modulo b, while the right hand side is congruent to -1 modulo b. Thus we have a contradiction. Thus, we must have $j = \ell$. It then follows that:

$$b^{i-j} - 1 \;=\; b^{k-\ell} - 1 \;\;\Rightarrow\;\; i - j \;=\; k - \ell \;\;\Rightarrow\;\; i = k.$$

Figure 2.5 A $\mathbb{Z}_{63}$-Valuation on K_6



Therefore, the edge labels induced by this function are distinct. Thus f is the required valuation. To achieve the bound described above, minimize these labels with $b = 2$. ■

The above labelling scheme is nice as it is easy to remember and compute. However, it is often undesirable as the labels on the vertices get quite large. Later, we will discuss ways of improving this bound.

Corollary 2.3.12 *[6] Every graph G has a $\mathbb{Z}_n$-valuation, for suitably large n. Thus, every G is a cyclic divisor of some circulant.*

Proof.

Suppose that $n(G) = p$. For $n \geq 2^p - 1$, a $\mathbb{Z}_n$-valuation was given for K_p in Theorem 2.3.11. Since our definition of valuation places no restriction on pairs of non-adjacent vertices, this same valuation will also work for any arbitrary graph of order p. Since any graph with a $\mathbb{Z}_n$-valuation is a cyclic divisor of an appropriately chosen circulant (i.e., one with order n and difference set equivalent to the set of induced edge labels) by Theorem 2.3.10, this completes the proof. ■

The bounds given above can be improved upon significantly with the introduction of perfect difference sets and Golomb rulers.

A set J of non-negative integers forms a *Golomb ruler* if and only if no integer can be expressed in two different ways as the difference of two members of J. Noting that $a-b = c-d$

Table 2.1 Lengths of Small Optimal Golomb Rulers

n	12	13	14	15	16	17	18	19	20	21	22	23
m	85	106	127	151	177	199	216	246	283	333	356	372

is equivalent to $a+d=b+c$, J is a Golomb ruler if and only if no integer can be expressed in two different ways as the sum of two members of J. For example, $\{0,1,3,5\}$ is *not* a Golomb ruler because $5-3=3-1$, and $\{0,1,2,3\}$ is *not* a Golomb ruler since $2-0=3-1$. In a Golomb ruler of cardinality p, there are $\binom{p}{2}$ distinct sums of distinct pairs and p sums of an element with itself, making $\binom{p+1}{2}$ distinct sums in all. Note that if J is a Golomb ruler and $m = \max J$, then the reflection $m - J := \{m-x : x \in J\}$ is also a Golomb ruler. Indeed, both sets have the same differences because $(m-x)-(m-y) = y-x$. However, the definition is usually given in terms of positive differences. But since $-(x-y) = y-x$, it follows that if all positive differences are distinct, then so are *all* differences.

The elements of a Golomb ruler are often called *marks* and its maximal element is often called its *length*. An n mark Golomb ruler with length m is *optimal* if and only if

i) there is no Golomb ruler with n marks of a shorter length and

ii) when the marks are put in ascending order, J equals or lexicographically precedes $m - J$.

The lengths of small optimal Golomb rulers are given in Table 2.1, while the marks of smaller Golomb rulers are given in Table 2.2. For each integer $n \geq 1$ let

$$Gol(n) := \min\{\max(J) : J \text{ is a Golomb ruler of } n \text{ marks}\}.$$

Theorem 2.3.13 *[6]* $Gol(n) \leq \dfrac{n^3-n^2+4}{2}$.

Table 2.2 Small Optimal Golomb Rulers

n												
1	0											
2	0	1										
3	0	1	3									
4	0	1	4	6								
5	0	1	4	9	11							
6	0	1	4	10	12	17						
7	0	1	4	10	18	23	25					
8	0	1	4	9	15	22	32	34				
9	0	3	9	17	19	32	39	43	44			
10	0	1	6	10	23	26	34	41	53	55		
11	0	1	4	13	28	33	47	54	64	70	72	
12	0	2	6	24	29	40	43	55	68	75	76	85

Proof.

We will prove by induction on n that there is a Golomb ruler J_n such that $|J_n| = n$. This is certainly the case for $n = 1$ since $\{0\}$ is a Golomb ruler. We wish to create J_{n+1} by adding some number x to J_n. Since there are no repeated pairwise sums from J_n, x will be inadmissible if and only if for some $a, b, c \in J_n$ we have $2x = a + b$ or $x + c = a + b$. In particular, x is forbidden to assume any of the following forms:

$$(a + b)/2 \qquad \text{or} \qquad a + b - c$$

The first form occurs in $\binom{n}{2}$ ways if a and b are distinct. There are an additional n exclusions of x being an existing member of J_n. The second form has $a + b$ occurring in $\binom{n}{2}$ ways if a and b are distinct. The choice of c may be assumed to be different from a and b since we have already excluded membership in J_n. Hence, c occurs in $n - 2$ ways. Now if $a = b$, then the second form reduces to $2a - c$ which occurs in $\binom{n}{2}$ ways. Thus, the number of inadmissible choices for x is at most:

$$\binom{n}{2} + n + \binom{n}{2}(n-2) + \binom{n}{2} \quad = \quad \binom{n}{2}n + n \quad = \quad \frac{n^3 - n^2 + 2n}{2}. \tag{2.1}$$

The bound given in the statement of the theorem takes on the following value for $n+1$:

$$\frac{(n+1)^3 - (n+1)^2 + 4}{2} \quad = \quad \frac{n^3 + 2n^2 + n + 6}{2}. \tag{2.2}$$

The bound in Equation 2.2 exceeds the count in Equation 2.1 by at least 1. Hence, it is possible to find a suitable x less than or equal to $\frac{(n+1)^3-(n+1)^2+4}{2}$ with which to augment J_n and form J_{n+1}. ■

Significantly better bounds can be obtained using the distribution of primes and difference sets.

A *perfect difference set* is a set $S = \{a_1, a_2, ..., a_{k+1}\} \pmod{n}$ such that every non-zero element can be uniquely expressed in the form $a_i - a_j \pmod{n}$. Note that this implies the existence of $2\binom{k+1}{2}$ distinct elements in addition to the zero element modulo n. Hence, it is necessary that $n = k^2 + k + 1$ for the existence of such a set. If k is a prime power, then a perfect difference set is known to exist [36]. Some small difference sets can be found in Table 2.3 as well as at the *La Jolla Repository of Difference Sets* which can be accessed at http://www.ccrwest.org/diffsets.

From the definition, it easily follows that a perfect difference set is a complete $\mathbb{Z}_{q^2+q+1}$-valuation on K_{q+1}. Thus, we can use this to improve the bounds given in Theorem 2.3.11. Note that while the bounds given by perfect difference sets are significantly smaller, it is often difficult to construct the required set. The bounds given in Theorem 2.3.11 are much easier to produce quickly.

While cyclic decompositions do give a convenient method of constructing a decomposition, they are restrictive in other ways.

Theorem 2.3.14 *[6] Let $H = (W, F)$ and G be graphs. If H has a cyclic G-decomposition, then H is a circulant.*

Table 2.3 Small Difference Sets

n	$k-1$												
7	3	0	1	3									
13	4	0	1	3	9								
21	5	0	3	4	9	11							
31	6	0	4	10	23	24	26						
57	8	0	1	6	15	22	26	45	55				
73	9	0	1	12	20	26	30	33	35	57			
91	10	0	2	6	7	18	21	31	54	63	71		
133	12	0	9	10	12	26	30	67	74	82	109	114	120

Proof.

Suppose that H admits a cyclic G-decomposition $\mathscr{D}$. Let f and A_0 be as in the definition of cyclic decomposition. Suppose that $n(H) = n$. Since f is a cyclic permutation of the n nodes of H, it follows that the nodes of H may be identified with the integers in $\mathbb{Z}_n$ in their cyclic order. More fully, select any node v_0 of A_0. Applying f to v_0 generates the orbit of v_0 with respect to f. Since f is cyclic as a permutation of $V(H)$, this orbit is all of $V(H)$. Denote $f^j(v_0)$ by v_j. Then the natural identification with $\mathbb{Z}_n$ is given by $j \to v_j$. In particular, we want to check that f acts as the successor function on the nodes after this identification:

$$f^k(j) \equiv f^k(v_j) = f^k(f^j(v_0)) = f^{k+j}(v_0) = v_{k+j} \equiv k + j$$

The blocks may be identified as $A_0, A_1, \dots, A_{n-1}$ where $A_j = f^j[A_0]$. Let

$$S := \{x - y : xy \in E(A_0)\}.$$

We claim H is isomorphic to the circulant $C_n(S)$. Suppose $ab \in E(H)$. Since $\mathscr{D}$ is a decomposition, ab belongs to some block A_j. Thus, there is an edge $xy \in E(A_0)$ such that $f^j[xy] = ab$, i.e., $f^j(x) = a$ and $f^j(y) = b$. In light of our identification of $V(H)$ with $\mathbb{Z}_n$, $a = x+j$ and $b = y+j$. Thus, $a-b = x-y \in S$, so $ab \in E(C_n(S))$. Hence, $E(H) \subseteq E(C_n(S))$.

Conversely, suppose $cd \in C_n(S)$. That means $c-d=s$ for some $s \in S$ and some pair of nodes c and d. Let x and y be nodes in A_0 such that $x-y=s$. That is, $x-y=c-d$, or equivalently, $c-x$ and $d-y$ share a common value j. Note then that $c=x+j$ and $d=y+j$. Recall that the action of f^j is just to add j. Thus, the edge cd is the image of the edge xy under f^j. Since cd is the image of an edge of H under the automorphism f^j, it follows that cd must also be an edge of H. Therefore, H and $C_n(S)$ are isomorphic. ■

Proposition 2.3.15 *[9] A cyclic G-decomposition of H is an $n(G)$-balanced decomposition.*

It should be noted that a cyclic decomposition is not necessarily simple or even λ-uniform. For instance, consider the cyclic P_3-decomposition of K_5 generated by the block 201. The blocks in this decomposition are 201, 312, 423, 034, and 140. Notice that 201 and 312 share two common nodes in K_5. However, 201 and 423 share only one common node. There will be some regularity in how the blocks intersect though.

Proposition 2.3.16 *Let $\mathscr{D}$ be a cyclic G-decomposition of H and let f be a cyclic permutation associated with $\mathscr{D}$. Let $G_0, G_1 \in \mathscr{D}$ be such that G_0 and G_1 share k common nodes in H. Then for all $\ell \in \mathbb{N}$, $f^\ell[G_0]$ and $f^\ell[G_1]$ share k common nodes in H.*

Proof.

Suppose that $v \in V(H)$ and $v \in V(G_0) \cap V(G_1)$. Let $\mathscr{D}$ be a cyclic G-decomposition of H and let f be the cyclic permutation associated with $\mathscr{D}$. By definition of cyclic decomposition, there exists a non-negative integer i such that $G_1 = f^i[G_0]$. Define $G_2 := f^\ell[G_0]$ and $G_3 := f^\ell[G_1] = f^{i+\ell}[G_0]$. Since G_0 and G_1 share the node v, there exists $w \in V(G_0)$ such that $f^i(w) = v$. However, by definition of G_2 and G_3, we must have $f^{i+\ell}(w) = f^\ell(v) \in V(G_2)$ and $f^\ell(v) \in V(G_3)$. Thus, there is a bijection between the shared nodes of G_0 and G_1 and the shared nodes of G_2 and G_3. Hence, if G_0 and G_1 share k common nodes in H, it follows that G_2 and G_3 share k common nodes. Since ℓ was chosen arbitrarily, it follows that this holds for all ℓ. ■

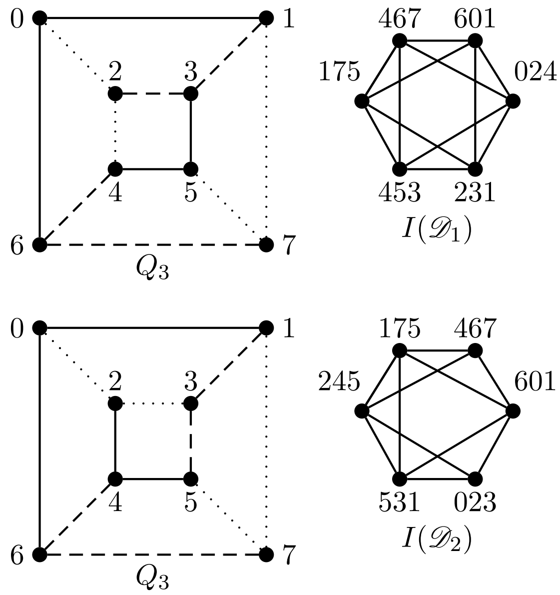
In the next chapter, we will look at how the existence of a projective plane can give us the desired decomposition. Note that a perfect difference set will induce a $\mathbb{Z}_{q^2+q+1}$-valuation on K_{q+1}. Thus, a perfect difference set will induce a cyclic K_{q+1}-decomposition of K_{q^2+q+1} by Theorem 2.3.10. We will show in Theorem 3.1.5 how this implies the existence of a projective plane.

2.4 Intersection Graphs

By definition, the blocks in a G-decomposition of H are edge-disjoint, but in general they share common nodes. The *intersection graph* $I(\mathscr{D})$ of the decomposition $\mathscr{D}$ has a vertex for each block of the partition and two blocks A and B are adjacent if and only if they share a common node in H. If k G-blocks share a common node in H, we say that the G-decomposition graph has a *local clique* of order k. There has been a tremendous amount of research on the chromatic number of a G-decomposition graph [7, 15, 16, 18, 19, 23, 24, 22, 28, 34, 45, 46, 47, 62, 63].

Example 2.4.1 *Let $H = Q_n$ and $G = K_{1,n}$. By placing the centers of our stars on the nodes of H with even parity, we partition all of the edges around these nodes, and thus all of the edges. This gives us the required decomposition, $\mathscr{D}$. Since Q_n is n-regular and bipartite, we could have equivalently placed the centers on the nodes with odd parity and achieved an isomorphic decomposition. Thus, the vertex set of $I(\mathscr{D})$ can be identified with the set of all n-tuples with entries in $\{0,1\}$ and even parity. The edges of $I(\mathscr{D})$ are the pairs of n-tuples which differ in exactly two positions. Note that the decomposition given in this example extends to all n-regular bipartite graphs. However, the intersection structure is harder to describe in most cases.*

Figure 2.6 Q_3 and Intersection Graphs of P_3-Decompositions



While line graphs and the decomposition described in Example 2.4.1 are unique up to isomorphism, this is generally not the case, even when taking a relatively simple host and prototype.

Example 2.4.2 *Let $H = Q_3$ and $G = P_3$. Consider the labelling of Q_3 given in the right of Figure 2.6. We present two P_3-decompositions of Q_3. $\mathscr{D}_1$: 467, 601, 024, 231, 175, 453 and $\mathscr{D}_2$: 467, 601, 023, 531, 175, 245. Note that the P_3-decomposition graphs are non-isomorphic as one is isomorphic to $C_6(1,2)$ and thus is regular of degree four and the other has vertices of degree three.*

In the previous example, we saw that a particular graph $C_6(1,2)$ could be expressed as a P_3-decomposition graph. This can be greatly generalized as a variation on a result by Erdős, Goodman, and Pósa [27].

Theorem 2.4.3 *Let K and G be graphs such that $n(G) \geq \Delta(K)$. Then there exist a graph H and a G-decomposition $\mathscr{D}$ of H such that $I(\mathscr{D}) \cong K$.*

Proof.

It suffices to give a construction of the required H. Let $V(K) = \{v_1, ..., v_{n(K)}\}$. For each $v_i \in V(K)$, we replace v_i by a copy of G, called G_i. Define $V(G_i) = \{u_{i,1}, ..., u_{i,n(G)}\}$. We construct H recursively, beginning with G_1. Let v_j be the vertex of K with the lowest index that is adjacent to v_1. Identify the nodes $u_{1,1}$ and $u_{j,1}$ as a single node in H. For each v_i adjacent to v_1, we identify the nodes $u_{i,1}$ and $u_{1,k}$ where k is the lowest index of a node that has not been identified. Note that since $n(G) \geq \Delta(K)$, this construction is well defined. Assume that the intersections for G_1,...,G_{i-1} are complete. For $v_iv_j \in E(K)$ where $j > i$, let $u_{i,k}$ be the first node of G_i that has not been adjoined. Similarly, we take $u_{j,\ell}$ be the first node of G_j that has not been adjoined. We then identify the nodes $u_{i,k}$ and $u_{j,\ell}$ as a single node in H. This completes the construction of H. ■

Example 2.4.4 *Consider the graphs K and G given in Figure 2.7. Let the block G_i represent the vertex v_i in K.*

- *Since v_1 is adjacent to v_2, v_3, v_4, and v_6 in K, we identify the following nodes: $u_{1,1} = u_{2,1}$, $u_{1,2} = u_{3,1}$, $u_{1,3} = u_{4,1}$, and $u_{1,4} = u_{6,1}$.*
- *Since v_2 is adjacent to v_1 and v_3 in K, we identify $u_{2,2} = u_{3,2}$.*
- *Since v_3 is adjacent to v_1, v_2, and v_4, we identify $u_{3,3} = u_{4,2}$.*
- *Since v_4 is adjacent to v_1, v_3, v_5, and v_6, we identify $u_{4,3} = u_{5,1}$ and $u_{4,4} = u_{6,2}$.*
- *Since v_5 is adjacent to v_4 and v_6, we identify $u_{5,2} = u_{6,3}$.*
- *The adjacency of v_6 has been established by previous nodes.*

Note that in Figure 2.7 we have only labelled nodes by their lowest index for simplicity.

It should be noted that while every graph can be represented as a G-decomposition graph, there are examples of graphs that cannot be represented as a line graph. Building on the earlier work of Krausz [58] as well as that of van Rooij and Wilf [71], Beineke [8] was

Figure 2.7 All Graphs Can Be Represented As Decomposition Graphs

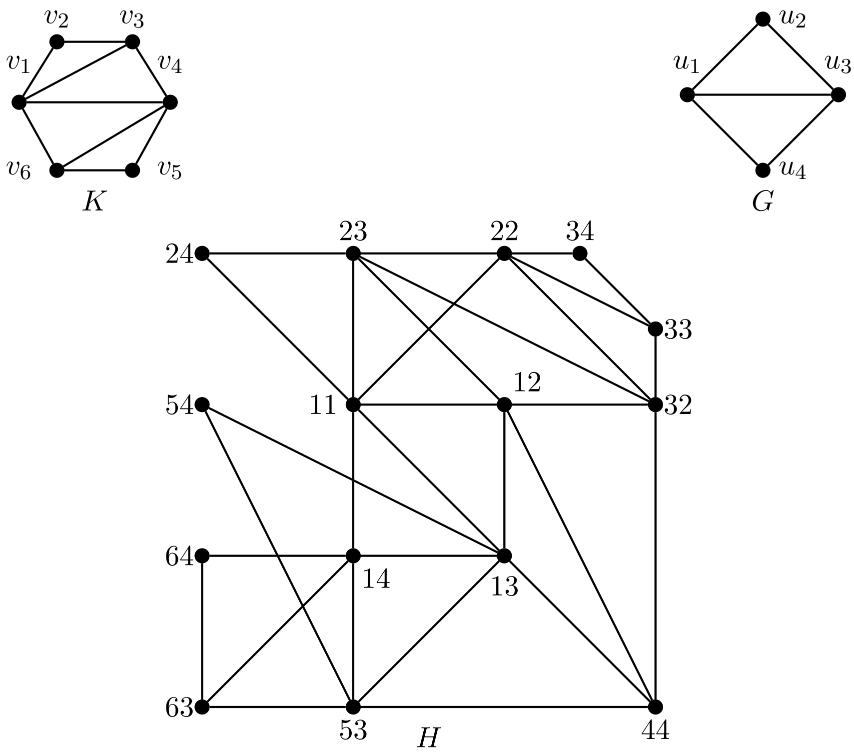
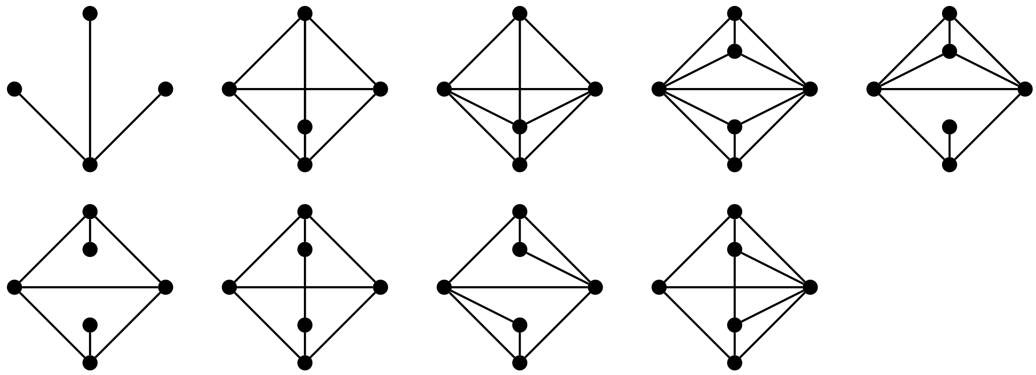


Figure 2.8 Graphs That Are Not Vertex Induced Subgraphs of Line Graphs



able to show that a graph can be represented as a line graph if and only if it contains none of the graphs listed in Figure 2.8 as a vertex induced subgraph.

This dissertation is primarily concerned with decomposition in which the decomposition graph is isomorphic to the host. One might wonder if there are decompositions in which the decomposition graph is isomorphic to the prototype. However, Theorem 2.4.3 makes the question trivial, as $n(G) \geq \Delta(G)$ for all G.

The proof given in Theorem 2.4.3 can be simplified a bit with the assumption that $n(G) \geq n(K)$. In this case, we say that the nodes $u_{i,j}$ and $u_{j,i}$ are identified if and only if $v_i v_j \in E(K)$. Given K, it is also of interest to determine the smallest G such that there exists an H and a G-decomposition $\mathscr{D}$ of H such that $I(\mathscr{D}) \cong K$. For example, if K contains cliques, then several blocks may share a single common node, and one can obtain the required intersection with a smaller prototype. This gives rise to a general principle.

Theorem 2.4.5 *Let K, G be graphs. For any $A \subseteq V(K)$, if K_A is triangle-free and $\Delta(K_A) > n(G)$ then there exists no H and $\mathscr{D}$, a G-decomposition of H, such that $I(\mathscr{D}) \cong K$.*

Proof.

Let $A \subseteq V(K)$ be such that K_A is triangle-free and $\Delta(K_A) > n(G)$. It suffices to show that there is no H and $\mathscr{D}$, a G-decomposition of H, such that $I(\mathscr{D}) \cong K_A$. Assume to the contrary that there is such an H and $\mathscr{D}$. Since $K_A \cong I(\mathscr{D})$ and K_A is triangle-free, it follows that no three blocks in $\mathscr{D}$ may share a common node in H. Let $v \in A$ be such that $deg_{K_A}(v) = \Delta(K_A)$ and take $B_v \in \mathscr{D}$ to be the G-block represented by v in $I(\mathscr{D})$. Thus, B_v must intersect $\Delta(K_A)$ other G-blocks. Since no three blocks may share a common node in H, it follows that $n(G) = n(B_v) \geq \Delta(K_A)$. However, $\Delta(K_A) > n(G)$ by hypothesis, hence a contradiction. Thus, it follows that there is no H and $\mathscr{D}$ such that $I(\mathscr{D}) \cong K_A$. From this it follows that there is no H and $\mathscr{D}$ such that $I(\mathscr{D}) \cong K$. ■

It would be nice to extend this notion to obtain a complete characterization of G-decomposition graphs in the same way that Krausz [58], van Rooij and Wilf [71], and Beineke [8] were able to completely characterize line graphs. However, their proof does not extend readily to G-decomposition graphs. In the proof, they note that a triangle in K must be represented by a triangle in H, arguing that any node adjacent to one edge of the triangle must be incident with another. However, in a G-decomposition graph, the corresponding triangle in $I(\mathscr{D})$ would then be represented by a series of G-blocks. Now any node incident with one of these G-blocks would not necessarily be incident with another, as each block may contain more than two nodes.

Proposition 2.4.6 *Let $\mathscr{D}$ be a G-decomposition of H, with $b = \frac{e(H)}{e(G)}$. If $n(G) > \frac{n(H)}{2}$, then $I(\mathscr{D}) \cong K_b$.*

Proof.

Since $n(G) > \frac{n(H)}{2}$, it follows that any two G-blocks in $\mathscr{D}$ must intersect. Since there are $b = \frac{e(H)}{e(G)}$ blocks, it follows that $I(\mathscr{D}) \cong K_b$. ■

When the decomposition is cyclic, we can say a bit more about the structure of the G-decomposition graph. Recall that in Theorem 2.3.14, we showed that if H admits a cyclic decomposition, then $H = C_n(S)$.

Theorem 2.4.7 *Let G be a simple graph and take f to be a $\mathbb{Z}_n$-valuation of G. Define $S = f^*(E(G))$ and $S' = f^*(G)$. If $\mathscr{D}$ is the cyclic decomposition of $H = C_n(S)$ induced by f, then $I(\mathscr{D}) \cong C_n(S')$.*

Proof.

Let $p = |V|$. Suppose that $f(V) =< f_1, f_2, ..., f_p >$ is the array representation of the labels of G. Since $\mathscr{D}$ is a cyclic decomposition, there is a G-block with 0 in the first position. Thus, we may assume without loss of generality that there is a G-block, $b_0 \in \mathscr{D}$, such that $b_0 :=< a_1, a_2, ..., a_p >$, where $a_1 := 0$. Note that b_0 will intersect blocks of the form $b_{a_j - a_i}$ for $1 \leq j \leq p$ and $1 \leq i \leq p$ as these blocks have a_j in the ith position. Thus, b_0 intersects blocks of the form b_k where $k = \pm(a_j - a_i)$. Note that $f(G) = |a_j - a_i|_n$ for $1 \leq i \leq j \leq p$. Since $\mathscr{D}$ is cyclic, it follows that b_a intersects with blocks of the form b_k where $k = a \pm (a_j - a_i)$. Label the vertices of $I(\mathscr{D})$ according to the index of their corresponding block. Clearly, there are n. If $a \in V(I(\mathscr{D}))$, then a intersects $a \pm k$ where $k \in S' = f^*(G)$. Thus, it follows that $I(\mathscr{D}) \cong C_n(S')$. ■

CHAPTER 3

AUTOMORPHIC DECOMPOSITIONS

Such papers as [23, 24, 44, 45, 46, 47] are concerned with whether the chromatic number of a G-decomposition graph can equal certain numbers. Thus, at least implicitly, these papers are concerned with the structure of a G-decomposition graph. We refine this problem further with the notion of automorphic decompositions.

Let $\mathscr{D}$ be a G-decomposition of H. We say that $\mathscr{D}$ is an *automorphic* G-decomposition if $I(\mathscr{D}) \cong H$. If such a decomposition exists, we say that G is an *automorphic divisor of* H. If G is an automorphic divisor of some graph, we say G is an *automorphic divisor*. If H admits an automorphic decomposition, then we say that H is an *automorphic host*.

3.1 Examples of Automorphic Decompositions

In this section, we give several examples of automorphic decompositions. Our first example comes from line graphs.

Proposition 3.1.1 *C_n has an automorphic P_2-decomposition and this decomposition is unique up to isomorphism.*

Proof.

Let $\mathscr{D}$ be a P_2-decomposition of C_n. By definition, $L(C_n) \cong I(\mathscr{D})$. By Proposition 1.1.9 we have $L(C_n) \cong C_n$. Thus by transitivity, $I(\mathscr{D}) \cong C_n$. Hence, $\mathscr{D}$ is an automorphic P_2-decomposition of C_n. Further, since the line graph is unique, this decomposition is unique up to isomorphism. ■

By relaxing our assumptions, a second trivial example follows.

Proposition 3.1.2 *Let $\mathcal{V}$ be an infinite set of isolated vertices. Define $H = (\cup_{i=2}^{\infty} P_i) \cup \mathcal{V}$. H has an automorphic P_2-decomposition.*

Proof.

Let $n \in \mathbb{N}$ be such that $n \geq 2$. Define $\mathscr{D}_n$ to be a P_2-decomposition of P_n. Note that $I(\mathscr{D}_n)$ is equivalent to finding the line graph of P_n. Thus by Proposition 1.1.9, we have that $I(\mathscr{D}_n) \cong P_{n-1}$ for $n \geq 2$. Let $\mathscr{D}$ be a P_2-decomposition of H. Note that for each path of length n in H, there will be a path of length $n-1$ in $I(\mathscr{D})$. Thus:

$$I(\mathscr{D}) \cong (\cup_{i=1}^{\infty} P_i) \cup \mathcal{V}.$$

However, since P_1 is an isolated vertex, we have that:

$$I(\mathscr{D}) \cong (\cup_{i=2}^{\infty} P_i) \cup \mathcal{V}.$$

Thus $\mathscr{D}$ is automorphic by definition. ■

Both of these examples are somewhat unsatisfying. The first example is unsatisfying because of the trivial nature of its construction. The second, because it violates several of our standing assumptions. In particular, the host is an infinite graph with isolated vertices. Thus we should seek out better examples.

Theorem 3.1.3 *G is a cyclic automorphic divisor of $C_n(S)$ if and only if there exists a closed $\mathbb{Z}_n$-valuation f on G such that $f^*(E(G)) = S$.*

Proof.

Recall that in Theorem 2.3.10 we showed that a graph G is a cyclic divisor of a circulant $H = C_n(S)$ if and only if there is a $\mathbb{Z}_n$-valuation f on G such that $f^*(E(G)) = S$. Further, we showed in Theorem 2.4.7 that the intersection graph generated by a G-decomposition will be isomorphic to $C_n(f^*(G))$. Note that:

$$C_n(f^*(G)) \cong C_n(f^*(E(G))) \Leftrightarrow f^*(G) \equiv f^*(E(G)) \pmod{n}.$$

But, $f^*(G) \equiv f^*(E(G)) \pmod{n}$ implies that f is a closed $\mathbb{Z}_n$-valuation of G. Thus, G is a cyclic automorphic divisor of $C_n(S)$ if and only if there exists a closed $\mathbb{Z}_n$-valuation f on G such that $f^*(E(G)) = S$. ■

This observation allows us to exploit several examples of classical valuations in order to show the existence of an automorphic decomposition.

Corollary 3.1.4 *Let G be a graph of size q.*

(i) If f is a graceful labelling of G, then G is an automorphic divisor of $C_n(S)$ where $n \geq 2q+1$ and $S = f^*(E(G))$.

(ii) If f is a ρ-valuation of G, then G is an automorphic divisor of K_{2q+1}.

Proof.

(i) In Proposition 2.1.5, we showed that if f is a graceful labelling on a graph of size q, then f is a closed $\mathbb{Z}_n$-valuation for all $n \geq 2q+1$. Applying this to Theorem 3.1.3 yields the desired result.

(ii) In Proposition 2.1.5, we showed that if f is a ρ-valuation on a graph of size q, then f is a closed $\mathbb{Z}_{2q+1}$-valuation. Note that every non-identity element of $\mathbb{Z}_{2q+1}$ must be used to valuate a graph of size q. Hence $f^*(E(G)) = \mathbb{Z}_{2q+1}$. Applying this to Theorem 3.1.3 yields the desired result. ■

In Proposition 2.1.6 and Proposition 2.1.7 we gave several examples of graphs that have graceful labellings and ρ-valuations. This gives us an infinite number of non-trivial examples of automorphic decompositions.

It should be noted that our examples are not limited to cyclic decompositions. While a construction is acceptable as a means of obtaining the required decomposition, we can also obtain a decomposition through the existence of a design.

Theorem 3.1.5 *For $n \geq 2$, K_{n^2+n+1} has an automorphic K_{n+1}-decomposition if and only if there exists a projective plane of order n.*

Proof.

Suppose that there is a projective plane of order n. A projective plane of order n has $n+1$ points on every line and $n+1$ lines passing through any point [73]. We equate the points of the projective plane to the nodes of K_{n^2+n+1} and the lines of the projective plane to the K_{n+1}-blocks. Since each edge of K_{n^2+n+1} is uniquely determined by a pair of nodes and each pair of points in the projective plane uniquely determines a line, it follows that no edge of K_{n^2+n+1} is contained in two K_{n+1}-blocks. Note that $n+1$ K_{n+1}-blocks intersect at each node of K_{n^2+n+1}, as $n+1$ lines pass through any point in the projective plane. Since each node has exactly n of its edges represented in a given block, it follows that for any given node, we have partitioned $n(n+1) = n^2+n$ of its incident edges (i.e., all of them). Thus, we have partitioned every edge, and no edge has been partitioned twice. Further, since any two lines in the projective plane share exactly one point, it follows that all of the blocks of this decomposition intersect. Hence, the G-decomposition graph is isomorphic to K_{n^2+n+1}. Ergo, the decomposition is automorphic by definition.

Conversely, suppose that $\mathscr{D}$ is an automorphic K_{n+1}-decomposition of $H = K_{n^2+n+1}$. Again, we define the nodes of our host as points and our K_{n+1}-blocks as lines. Thus, we need only confirm that the properties of a projective plane hold:

(i) The fact that given any two points (nodes) there is exactly one line (block) containing them follows from the fact that $\mathscr{D}$ is an edge decomposition.

(ii) Since $\mathscr{D}$ is an automorphic K_{n+1}-decomposition, it follows that given any two K_{n+1}-blocks, there is at least one point common to both. If these blocks shared at least two common nodes, this would be a violation of (i). Thus, every two lines (blocks) intersect in exactly one point (node).

(iii) Finally, we must show that H contains a set of four nodes such that no three of them lie in a common block. It suffices to construct the required set of nodes. For all $x, y \in V(H)$, define B_{xy} to be the unique block that contains x and y. Let $a, b, c \in V(H)$ be such that $c \notin B_{ab}$. This implies that B_{ab}, B_{ac}, and B_{bc} are distinct.

Since $n \geq 2$, it follows that each of these K_{n+1}-blocks contain at least three nodes. Let $d \in B_{ab}$ and $e \in B_{ac}$ be such that $d, e \notin \{a, b, c\}$. By (ii), there is a unique node f common to both B_{cd} and B_{be}. We claim that $\{a, b, c, f\}$ is the required set of nodes. Note that by hypothesis a, b, c are not elements of the same block. Thus, we need only show that $f \notin B_{ab}, B_{ac}, B_{bc}$. Suppose to the contrary that $f \in B_{ab}$. Since two distinct blocks share exactly one common node, it follows that $B_{be} = B_{ab}$. Thus $e \in B_{ab}$. We also have that $e \in B_{ac}$ and since two distinct blocks share exactly one common node, it follows that either $e = a$ or $B_{ac} = B_{ab}$. Because either possibility violates our hypothesis, it follows that $f \notin B_{ab}$. A similar argument can be used to show that $f \notin B_{ac}$ and $f \notin B_{bc}$. Thus we have constructed the required set. ■

We suspect that it is difficult to find a non-circulant host that admits an automorphic decomposition. First, for arbitrary graphs G and H it is generally hard to determine if $G|H$. Second, the problem of determining whether $I(\mathscr{D}) \cong H$ is a notoriously difficult one. Thus, it may be tempting to presume that only circulants admit automorphic decompositions. We now present a counterexample to this presumption.

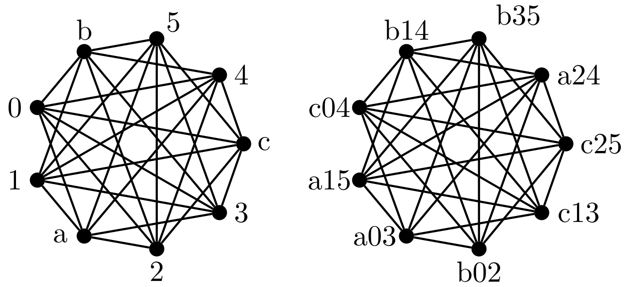
Proposition 3.1.6 *If $H \cong \overline{K_3} \vee C_6(2,3)$, then H is not isomorphic to any circulant.*

Proof.

Note that $H \cong \overline{K_3 \cup \overline{C_6(2,3)}}$ by Proposition 1.1.7. Since $\overline{C_6(2,3)} \cong C_6(1) \cong C_6$ by Proposition 2.2.2, it follows that $H \cong \overline{K_3 \cup C_6}$. Equivalently, we must have that $\overline{H} \cong K_3 \cup C_6$. Since all the connected components of a circulant must have the same order by Proposition 2.2.2, it follows that $\overline{H}$ is not isomorphic to any circulant. Since the complement of a circulant is likewise a circulant by Proposition 2.2.2, it follows that H is not isomorphic to any circulant. ■

Theorem 3.1.7 $H \cong \overline{K_3} \vee C_6(2,3)$ *admits an automorphic K_3-decomposition.*

Figure 3.1 $\overline{K_3} \vee C_6(2,3)$ and a K_3-Decomposition Graph



Proof.

In H, label the nodes of $\overline{K_3}$ with a, b, and c. Label the nodes of $C_6(2,3)$ in the standard way (see Figure 3.1). Let $\mathscr{D}$ be a K_3-decomposition with the following blocks: $a03$, $b14$, $c25$, $a24$, $b02$, $c04$, $a15$, $b35$, and $c13$. All of the edges generated by the join are partitioned, as each letter is in a K_3-block with each number. The edges of length three in $C_6(2,3)$ are partitioned by the blocks $a03$, $b14$, and $c25$. The edges of length two are partitioned by the blocks $a24$, $b02$, $c04$, $a15$, $b35$, and $c13$. As any K_3-decomposition is simple, it follows that no edge is in two distinct blocks. Thus, this is a valid K_3-decomposition of H.

It remains to show that this decomposition is automorphic. Since $a03$, $b14$, and $c25$ are mutually disjoint K_3-blocks, we let these be represented by the vertices of $\overline{K_3}$. Further, each of these blocks share a node with each of the remaining blocks. We allow the blocks $a24$, $b02$, $c04$, $a15$, $b35$, and $c13$ to be represented by the vertices of $C_6(2,3)$. If two of these blocks share a common letter, then there is edge of length three between them. The blocks with an even number form one of the induced three cycles in H. Similarly, the second three cycle is formed from the blocks with an odd number. Thus $I(\mathscr{D}) \cong H$, hence $\mathscr{D}$ is automorphic by definition. ■

We suspect that similar construction methods may be employed to find other non-circulant automorphic hosts, perhaps even an infinite class of non-circulant automorphic hosts. Unfortunately, it is unclear how to generalize the construction.

3.2 Necessary Conditions

Recall that we stated several necessary conditions for the existence of a G-decomposition of H in Theorem 2.3.7. We can state stronger necessary conditions for the existence of an automorphic G-decomposition of H. These will be used to help study graphs that may host an automorphic decomposition.

Theorem 3.2.1 *In addition to the conditions of Theorem 2.3.7, the following are necessary conditions for the existence of an automorphic G-decomposition of H:*

(i) $e(H) = n(H)e(G)$.

(ii) $\overline{d}(H) = 2e(G)$.

(iii) $\delta(H) \le 2e(G) \le \Delta(H)$.

(iv) $\alpha(H)n(G) \le n(H)$.

(v) For any node $v \in V(H)$, at most $\omega(H)$ G-blocks meet at v.

(vi) $\Delta(H) \le \omega(H)\Delta(G)$.

(vii) The average number of G-blocks that meet at any node of H is $n(G)$.

(viii)

$$n(G) \le \min\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\} \le \max\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\} \le \chi(H).$$

(ix) There exist non-negative integers $d_{\delta(G)}, ..., d_{\omega(H)\Delta(G)}$ *such that:*

$$\sum_{i=\delta(G)}^{\omega(H)\Delta(G)} d_i = n(H) \quad \textit{and} \quad \sum_{i=\delta(G)}^{\omega(H)\Delta(G)} id_i = 2e(G)n(H).$$

(x) There exist non-negative integers $b_1, ..., b_{\omega(H)}$ *such that:*

$$\sum_{i=1}^{\omega(H)} b_i = n(H) \quad \textit{and} \quad \sum_{i=1}^{\omega(H)} ib_i = n(G)n(H).$$

(xi) For all $A \subseteq V(H)$ *such that* H_A *is triangle-free, we require that* $n(G) \geq \Delta(H_A)$.

(xii) If H *is triangle-free, we require that* $\chi(H) \geq \omega(H) \geq n(G) \geq \Delta(H)$.

(xiii) If H *is triangle-free and not an odd cycle, we require that* $\chi(H) = \omega(H) = n(G) = \Delta(H)$ *and* $\Delta(G) = 1$.

Proof.

(i) Let $\mathscr{D}$ be a G-decomposition of H. Note that by definition of $I(\mathscr{D})$, $n(I(\mathscr{D}))$ is the number of G-blocks in $\mathscr{D}$. By Theorem 2.3.7, we have that:

$$n(I(\mathscr{D})) = \frac{e(H)}{e(G)}.$$

For $\mathscr{D}$ to be automorphic, we must have that $n(I(\mathscr{D})) = n(H)$. It follows that we must have that:

$$n(H) = \frac{e(H)}{e(G)} \Leftrightarrow e(G) = \frac{e(H)}{n(H)} \Leftrightarrow e(H) = e(G)n(H).$$

(ii) By Proposition 1.1.2 we have that:

$$\overline{d}(H) = \frac{1}{n(H)} \sum_{v \in V(H)} d(v) = \frac{2e(H)}{n(H)}. \tag{3.1}$$

But, if H has an automorphic decomposition with respect to G, it follows from (i), that Equation 3.1 is equivalent to:

$$\overline{d}(H) = \frac{2e(H)}{n(H)} = 2e(G).$$

(iii) From Proposition 1.1.2, we have that $\delta(H) \leq \overline{d}(H) \leq \Delta(H)$. Applying this to (ii), yields the desired result.

(iv) Let $\mathscr{D}$ be a G-decomposition of H. Since $\mathscr{D}$ is automorphic, $I(\mathscr{D})$ must have an independent set of size $\alpha(H)$. Thus, we must have $\alpha(H)$ independent G-blocks. Each block requires $n(G)$ distinct nodes, so we must have at least $\alpha(H)n(G)$ distinct nodes in H. Since H only has $n(H)$ nodes, we must have that $\alpha(H)n(G) \leq n(H)$.

(v) Note that if more than $\omega(H)$ G-blocks meet at a given node of H, then these blocks form a local clique of order greater than $\omega(H)$ in $I(\mathscr{D})$. This would imply that $\omega(I(\mathscr{D})) > \omega(H)$, a contradiction. Thus, at most $\omega(H)$ G-blocks meet at any node of H.

(vi) By (v) at most $\omega(H)$ blocks meet at any node of H. Since each of these blocks contribute at most $\Delta(G)$ edges, we must have that $\Delta(H) \leq \omega(H)\Delta(G)$.

(vii) Let $\overline{b}$ be the average number of G-blocks that meet at any given node of H. Each block covers $n(G)$ distinct nodes of H. Further, the total number of blocks is $n(H)$ by (i). Thus we must have:

$$\frac{n(H)\overline{b}}{n(G)} = n(H)$$
$$\Rightarrow \overline{b} = n(G).$$

(viii) Note that $n(G) \leq \frac{n(H)}{\alpha(H)}$ by (iv). Further, $\omega(H)$ is the maximum number of G-blocks at any node of H by (v), while $n(G)$ gives the average number of blocks at any node of H by (vii). Thus, we must have that $n(G) \leq \omega(H)$. Ergo, $n(G) \leq \min\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\}$. Clearly we have that:

$$\min\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\} \leq \max\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\}.$$

From Proposition 1.1.4, we have that $\chi(H) \geq \omega(H)$ and $\chi(H) \geq \frac{n(H)}{\alpha(H)}$. Thus, we must have that $\chi(H) \geq \max\{\omega(H), \frac{n(H)}{\alpha(H)}\}$. Hence, from transitivity, we have that:

$$n(G) \leq \min\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\} \leq \max\left\{\omega(H), \frac{n(H)}{\alpha(H)}\right\} \leq \chi(H).$$

(ix) Let d_i be the number of nodes of H which have degree i. Note that if G is an automorphic divisor of H, we must have that $\delta(H) \geq \delta(G)$ by Theorem 2.3.7 and $\Delta(H) \leq \omega(H)\Delta(G)$ by (vi). Note that the total number of nodes of H is $n(H)$ by definition. Thus we have:

$$\sum_{i=\delta(G)}^{\omega(H)\Delta(G)} d_i = n(H).$$

Since there are d_i nodes of degree i in H and each contributes i edges to the edge count of H, we must have:

$$\sum_{i=\delta(G)}^{\omega(H)\Delta(G)} id_i = 2e(H) = 2e(G)n(H).$$

(x) Let b_i be the number of nodes of H that have i G-blocks meeting at that node. Note that at most $\omega(H)$ blocks meet at any given node of H by (v). Since the total number of nodes of H is $n(H)$ by definition, we must have:

$$\sum_{i=1}^{\omega(H)} b_i = n(H)$$

By (vii), the average number of blocks that meet at a node of H is given by $n(G)$. Each of the b_i nodes contribute i blocks to this average. Thus we must have:

$$\sum_{i=1}^{\omega(H)} ib_i = n(G)n(H).$$

(xi) Note that in an automorphic decomposition, $I(\mathscr{D}) \cong H$ and apply Theorem 2.4.5.

(xii) If H is triangle-free, then take $A = V(H)$ and apply (xi) to yield $n(G) \geq \Delta(H)$. Applying (viii) gives us:

$$\chi(H) \geq \omega(H) \geq n(G) \geq \Delta(H).$$

(xiii) If H is triangle-free, then applying (xii) yields:

$$\chi(H) \geq \omega(H) \geq n(G) \geq \Delta(H).$$

Since H is triangle-free, it cannot be a complete graph. If we also have that H is not an odd cycle, then we have $\chi(H) \leq \Delta(H)$ by Brooks' Theorem (Proposition 1.1.4). Thus:

$$\chi(H) \geq \omega(H) \geq n(G) \geq \Delta(H) \geq \chi(H)$$

$$\Rightarrow \chi(H) = \omega(H) = n(G) = \Delta(H).$$

From (v), we have that at most $\omega(H)$ G-blocks meet at any node in H. However, (vii) implies that on average $n(G)$ G-blocks meet at any node in H. Since $\omega(H) = n(G)$, we must have that exactly $n(G)$ G-blocks meet at any node in H. Since $\Delta(H) = n(G)$, we must have that each of these G-blocks cover exactly one edge. Thus $\Delta(G) = 1$.

■

Remark 3.2.2 *From Theorem 3.2.1, we must have that the number of blocks in an automorphic decomposition is equal to the number of nodes in H. This implies that the underlying design must be symmetric. Thus, any necessary conditions for the existence of a symmetric design could be applied to the theory of automorphic decompositions.*

Note that West [75] considers $\omega(H) \leq \chi(H)$ to be a "bad bound," while he considers $n(H) \leq \chi(H)\alpha(H)$ to be substantially better. Hence, we should expect that $n(G) \leq \omega(H)$ to be a tighter bound on the existence of an automorphic decomposition. In the next chapter, we will study the case where $n(G) = \chi(H)$.

Theorem 3.2.1 allows us to show that several families of graphs do not have automorphic decompositions.

Corollary 3.2.3 *(i) If T is any tree, then T is not an automorphic host.*

(ii) If H is a d-regular graph, then H is not an automorphic host when d is odd.

(iii) K_n *is not an automorphic host when* n *is even.*

Proof.

(i) Let G be a prototype such that $e(G) = q$. Note that if $n(T) = n$, then $e(T) = n - 1$. Theorem 3.2.1 implies that for the existence of an automorphic G-decomposition, we must have that $n = \frac{n-1}{q}$. This implies that $q = \frac{n-1}{n}$. However, this is impossible as $\frac{n-1}{n} < 1$ for all $n \in \mathbb{Z}\backslash\{0\}$.

(ii) The degree d of a regular graph is equal to its average degree. By Theorem 3.2.1, d is even if H hosts an automorphic decomposition.

(iii) Note that since K_n is a $(n-1)$-regular graph, we must have that $n - 1$ is even, by (ii). Equivalently, we must have that n is odd.

■

Theorem 3.2.4 *Let* $H = (V, E)$ *be a finite graph.* H *has an automorphic* P_2*-decomposition if and only if* H *is the disjoint union of cycles.*

Proof.

If H is a disjoint union of cycles, then the P_2-automorphic decomposition of H is given by Proposition 3.1.1.

Conversely, suppose $\mathscr{D}$ is an automorphic P_2-decomposition of H. By Theorem 3.2.1, we have that $n(H) = e(H) = n$ and that $\overline{d}(H) = 2$. If H is a 2-regular graph, then H is a disjoint union of cycles by Proposition 1.1.3. Thus, we need only show that H has no node of degree one. Assume to the contrary that H has a node of degree one, call this node u. Let $\mathcal{C}$ be the connected component of H that contains u. Since $\sum_{a \in V(\mathcal{C})} deg_{\mathcal{C}}(a)$ is even, it follows that there exists $v \in V(H)$ such that $deg(v) \geq 3$. Since u and v are in the same connected component of H, it follows that there must be a shortest path between the two, say P_k. Without loss of generality, assume that this is the longest path between a node of degree one and a node of degree at least three. By definition, $I(\mathscr{D}) \cong H$. Thus, $I(\mathscr{D})$ must contain an induced subgraph that is isomorphic to P_k. However, to obtain a P_k in an intersection graph generated by a P_2-decomposition, we must have a P_{k+1} in H. However, this contradicts the assumption that P_k was the longest path between a node of degree one and a node of degree three. Hence, H must be a 2-regular graph. Thus by Proposition 1.1.3, H is a disjoint union of cycles. ■

Corollary 3.2.5 *Let H be a bipartite graph. H is an automorphic host if and only if H is the disjoint union of even cycles.*

Proof.

If H is the disjoint union of even cycles, then it has an automorphic decomposition with respect to P_2 by Theorem 3.2.4.

Suppose that H is a connected bipartite graph with an automorphic decomposition with respect to some prototype G (call this decomposition $\mathscr{D}$). Since H is bipartite, it follows that H is 2-colorable. For H to have an automorphic decomposition with respect to G, we must have that $n(G) \leq 2$ by Theorem 3.2.1. Since there is only one graph of order two that has no isolated vertices, it follows that $G \cong P_2$ [37]. Thus by Theorem 3.2.4, H must be the disjoint union of cycles. Since H is bipartite, it cannot contain any odd cycles [75]. Thus, H must be the disjoint union of even cycles. ■

In the next chapter, we will more thoroughly examine the question of which graphs can host an automorphic decomposition.

CHAPTER 4

AUTOMORPHIC HOSTS

In the previous chapter, we outlined several necessary conditions for the existence of an automorphic decomposition in Theorem 3.2.1. These conditions allowed us to show that certain graphs cannot host an automorphic decomposition. For instance, we showed in Theorem 3.2.4 that only a disjoint union of cycles can host an automorphic P_2-decomposition. As a corollary, we showed that the only bipartite graphs that can host an automorphic P_2-decomposition are disjoint unions of even cycles.

The goal of this chapter is to extend the study of graphs which may host an automorphic decomposition.

4.1 Simple Automorphic Decompositions

In order to facilitate our study of automorphic hosts, we often require additional hypotheses. For instance, in this section we place the additional restriction that the decomposition is *simple*, i.e., given any two G-blocks in the decomposition, these blocks share at most one common node in H. If $\mathscr{D}$ is a simple G-decomposition of H such that $I(\mathscr{D}) \cong H$, then we say that $\mathscr{D}$ is a *simple automorphic decomposition.* If such a decomposition exists, we say that G is a *simple automorphic divisor of* H. If G is a simple automorphic divisor of some graph, we say G is a *simple automorphic divisor.* If H admits a simple automorphic decomposition, we say that H is a *simple automorphic host.*

Theorem 4.1.1 *Let H and G be simple graphs where G is d-regular and of order p. If G is a simple automorphic divisor of H, then it follows that H is pd-regular.*

Proof.

Note that since G is a simple automorphic divisor of H, then any two G blocks in the decomposition $\mathscr{D}$ share at most one common node in H. For convenience of notation, let $n(H) = n$.

Further note that since G is d regular and $G|H$, then for all $v \in V(H)$, $d|deg_H(v)$ by Theorem 2.3.7. Let the degree sequence of H be given by $dseq(H) = [d_1, ..., d_n]$. Since $\mathscr{D}$ is an automorphic decomposition, it follows that $I(\mathscr{D}) \cong H$. This implies that they share the same degree sequence. Thus, for each i there exists a G-block $G_i \in \mathscr{D}$ that intersects exactly d_i other G-blocks.

Further, for each node v_i of degree d_i in H, exactly $\frac{d_i}{d}$ G-blocks meet at v_i as v_i is of degree d_i and each G-block partitions exactly d of those edges. Thus, for any G-block containing v_i, it intersects exactly $d_i^* := \frac{d_i}{d} - 1$ other G-blocks at v_i. Let $T(H)$ be the sequence of these transformed values, d_i^*. We intend to label the nodes of a given G-block with these values. Note that the label on a node of a given G-block is exactly the number of other blocks that share that same node in H. Further, since the decomposition is simple, the sum of these values gives the degree of the corresponding vertex in $I(\mathscr{D})$.

Listing the n G-blocks of the automorphic decomposition of H, we label the nodes of each G-block with the number of other G-blocks that meet at that particular node. Since G is a simple automorphic divisor, it follows that any two blocks share at most one common node in H. Thus, if G_i is the G-block representing a node of degree d_i in H, then the sum of the labels on the nodes of G_i must be d_i. Further note that these labels must come from $T(H)$. Also, for any given node v_i of degree d_i, exactly $d_i^* + 1$ G-blocks meet at v_i. Thus, each d_i^* appears as a node label exactly $d_i^* + 1$ times.

Suppose that for some $rd \in dseq(H)$, there exists no $sd \in dseq(H)$ such that $s < r$. This is equivalent to assuming that for some $r - 1 \in T(H)$ there exists no $s - 1 \in T(H)$ such that $s < r$. Note that if $r > p$, then $\overline{d}(H) > pd = 2e(G)$, contrary to G being an automorphic divisor of H. If $r = p$, then there are no nodes of H which have degree lower than pd. Since the average degree of H must be pd, it follows that H is pd-regular. Thus, we may assume that $r < p$.

If $r = 1$, then it vacuously holds that there is no $sd \in dseq(H)$ such that $s < r$, as we do not allow hosts with isolated nodes. Further, it vacuously holds that for all $s < r$, all of the $s - 1$ labels are allocated to the blocks representing a node of degree sd.

Suppose that there are k nodes in H that are of degree rd. This means that $r-1$ appears in $T(H)$ exactly k times. Let $G_1, ..., G_k$ be the G-blocks represented by the vertices of degree rd in $I(\mathscr{D})$. Note that we have exactly kr copies of $r-1$ with which to label the nodes of $G_1, ..., G_k$. Further, we have assumed that no node may have a label smaller than $r-1$.

Let $i \in \{1, ..., k\}$. Suppose that the nodes of G_i are labelled with at most $r-1$ copies of $r-1$. Thus, if $sum(G_i)$ is the sum of these labels, we have that:

$$sum(G_i) \geq (r-1)^2 + (p-r+1)r = r(p-1) + 1 > rd.$$

However, the labels on G_i should sum to exactly rd, a contradiction. Further, if we label the nodes of G_i with more than r copies of $r-1$, then there is $j \in \{1, ..., k\}$ such that G_j has at most $r-1$ copies of $r-1$ by the Pigeon Hole Principle. Thus for all $i \in \{1, ..., k\}$, G_i has exactly r nodes labelled with $r-1$. The remaining nodes of G_i must all be labelled at least r. This implies that:

$$rd = sum(G_i) \geq r(r-1) + (p-r)r = r(p-1) \geq rd.$$

Note that for equality to hold throughout, the remaining labels on G_i must all be r.

Suppose that $G_{k+1}, ..., G_{k+t}$ are represented by the vertices of degree $(r+1)d$ in $I(\mathscr{D})$. This means that r appears in $T(H)$ exactly t times. Note that we have exactly $t(r+1)$ copies of r with which to label the nodes of $G_{k+1}, ..., G_{k+t}$. Since there are no labels smaller than $r-1$ and all of the $r-1$ labels have been allocated to $G_1, ..., G_k$, it follows that the labels on $G_{k+1}, ..., G_{k+t}$ must be at least r.

Let $j \in \{k+1, ..., k+t\}$. Suppose that the nodes of G_j are labelled with at most r copies of r. This implies that:

$$sum(G_j) \geq r^2 + (p-r)(r+1) = r^2 + (p-1+1-r)(r+1)$$

$$= r^2 + (p-1)(r+1) - r^2 + 1 = (p-1)(r+1) + 1 > d(r+1).$$

However, the labels on G_j must sum to exactly $d(r+1)$, a contradiction. If G_j has more than $r+1$ copies of r, then by the Pigeon Hole Principle, there is $\ell \in \{k+1, ..., k+t\}$ such that G_ℓ has at most r copies of r, a contradiction. Thus, G_j has exactly $r+1$ nodes labelled with r. Since this holds for all $j \in \{k+1, ..., k+t\}$, we have allocated all of the $t(r+1)$ labels of r. Thus, we have no labels of r with which to label the remaining nodes of G_i. Hence, we have a contradiction.

Thus for all $r < p$, H can have no nodes of degree rd. Since $\overline{d}(H) = pd$ by Theorem 3.2.1, it follows that H is pd-regular. ■

Corollary 4.1.2 *Let H be a simple graph. If K_p is an automorphic divisor of H, then H is $p(p-1)$-regular.*

Proof.

Note that any K_p-decomposition is simple by Proposition 2.3.6. Further, K_p is $(p-1)$-regular. Applying the result from Theorem 4.1.1, we obtain that H is $p(p-1)$-regular. ■

While the existence of simple automorphic K_p-decompositions is guaranteed by Theorem 2.3.11, it may be difficult to find examples of simple automorphic divisors that are not complete. Many of our examples are based on cyclic decompositions, and Proposition 2.3.16 shows that the blocks in a cyclic decomposition will often intersect multiple times. As such, we conjecture that K_p is the only simple automorphic divisor. Progress towards a proof of this conjecture is given in the next corollary.

Corollary 4.1.3 *Let H and G be simple graphs such that G is a d-regular graph of order p. If H admits a simple automorphic G-decomposition, then $G \cong K_p$.*

Proof.

Since G is d-regular and of order p and since H admits a simple automorphic G-decomposition, it follows that H is pd-regular by Theorem 4.1.1. Thus, exactly p G-blocks meet at any given node of H. Note that any G-block in $G_i \in \mathscr{D}$ is represented by a vertex of degree pd in $I(\mathscr{D})$. Thus, for each node of G_i, exactly $p-1$ other G-blocks share that same node in H. Since $\mathscr{D}$ is simple, these blocks are all distinct. Ergo, exactly $p(p-1)$ distinct G-blocks intersect G_i. However, G_i is represented by a vertex of degree pd, hence exactly pd G-blocks must intersect G_i. This implies that $pd = p(p-1)$, or equivalently, $d = p-1$. Since the degree of regularity is one less than the order of G, it follows that $G \cong K_p$. ■

We can also give an additional necessary condition for hosts with an induced subgraph that satisfies certain properties.

Theorem 4.1.4 *Let H and G be graphs. Let $A \subseteq V(H)$. If H_A is triangle-free and G is a simple automorphic divisor of H, then it follows that $n(H) \geq n(H_A)n(G) - e(H_A)$.*

Proof.

Let $\mathscr{D}$ be a simple automorphic G-decomposition of H. It follows that there is a set of $n(H_A)$ G-blocks that represent H_A in $I(\mathscr{D})$. Let $v \in V(H_A)$ and let B_v be the G-block represented by v in the G-decomposition graph. Suppose that $deg_{H_A}(v) = d_v$. Note that B_v has $n(G)$ distinct nodes. Further, it is adjacent to d_v distinct G-blocks. Since G is a simple automorphic divisor, it follows that each of these blocks share exactly one node with B_v. Also, no two of these blocks share the same common node with B_v, as this would form a triangle in the intersection. Thus, there are $n(G) - d_v$ nodes that are unique to B_v. Ergo, the number of nodes in H_A that are in exactly one G-block is given by:

$$\sum_{v \in V(H_A)} (n(G) - d_v) = n(H_A)n(G) - 2e(H_A).$$

Since this is a simple decomposition, then any two blocks share at most one common node. Further, since H_A is triangle-free, then no three blocks representing H_A share a common node. Thus, the number of nodes that are shared is exactly the number of times the blocks intersect. However, each intersection represents an edge in H_A. Thus, the number of shared nodes is $e(H_A)$. Hence, the number of distinct nodes used to represent H_A is given by:

$$n(H_A)n(G) - 2e(H_A) + e(H_A) = n(H_A)n(G) - e(H_A).$$

Since the number of nodes in H must be at least the number of nodes used to represent one of its subgraphs, we must have that:

$$n(H) \geq n(H_A)n(G) - e(H_A).$$

■

Note that if we take H_A to be a maximum independent set of nodes (which is vacuously triangle-free), then $n(H_A) = \alpha(H)$ and $e(H_A) = 0$. Thus, Theorem 4.1.4 gives the inequality $n(H) \geq \alpha(H)n(G)$, the same result found in Theorem 3.2.1. Further note, for our argument, it suffices that the blocks that represent the induced subgraph share at most one common node in H. Thus, we may weaken our hypothesis from that of a simple decomposition.

Theorem 4.1.4 also allows us to show that additional classes of graphs are not automorphic hosts. In particular, we can rule out a possible extension of Theorem 3.1.7. Observe that $H \cong \overline{C_4 \cup C_5}$ meets the divisibility requirements for an automorphic K_3-decomposition. However, $\overline{C_4 \cup C_5}$ has a vertex induced subgraph isomorphic to C_5. Thus by Theorem 4.1.4:

$$9 = n(H) \geq n(G)n(C_5) - e(C_5) = 15 - 5 = 10.$$

Since $9 < 10$, we have a contradiction. Thus, H does not admit an automorphic K_3-decomposition.

4.2 Fully Automorphic Decompositions

Another profitable assumption is to assume that $\chi(H) = n(G)$. Note that if we assume that H is triangle-free and not an odd cycle, then we have $\chi(H) = n(G)$ as a consequence of Theorem 3.2.1. However, this also forced $\Delta(G) = 1$, a triviality that we wish to avoid.

Definition 4.2.1 *Let $\mathscr{D}$ be a G-decomposition of H. We say that $\mathscr{D}$ is a* fully automorphic G-decomposition *of H if $n(G) = \chi(H)$ and $I(\mathscr{D}) \cong H$. In this case we say that G is a* fully automorphic divisor *of H and that H is a* fully automorphic host.

We begin by listing necessary conditions for the existence of a fully automorphic decomposition.

Theorem 4.2.2 *Suppose that $\mathscr{D}$ is a fully automorphic G-decomposition of H. In addition to the conditions of Theorem 2.3.7 and Theorem 3.2.1, we require the following:*

(i) $n(G) = \omega(H) = \frac{n(H)}{\alpha(H)} = \chi(H)$.

(ii) In $\mathscr{D}$ there are exactly $n(G)$ G-blocks meeting at any given node of H.

(iii) Each color class of H must contain $\alpha(H)$ nodes.

(iv) If G is not the disjoint union of P_2's, then $\delta(H) \geq n(G) + 1$.

(v) $2e(H) \leq n(H)(n(H) - \alpha(H))$.

(vi) $\overline{d}(H) < \delta(H)\chi(H)$.

(vii) $2e(G) \leq n(H) - \alpha(H) = \alpha(H)(n(G) - 1)$.

Proof.

(i) Note that $n(G) \leq \omega(H) \leq \chi(H)$ and $n(G) \leq \frac{n(H)}{\alpha(H)} \leq \chi(H)$ by Theorem 3.2.1. Further, in a fully automorphic G-decomposition of H we have that $n(G) = \chi(H)$. From this it follows that:

$$n(G) = \omega(H) = \frac{n(H)}{\alpha(H)} = \chi(H).$$

(ii) By Theorem 3.2.1, we have that at most $\chi(H)$ G-blocks meet at a given node of H. On average, $n(G)$ blocks meet at a given node of H by Theorem 3.2.1. In a fully automorphic decomposition, we have $n(G) = \omega(H)$ by (i). Thus, there are no nodes of H that have more than the average number of blocks. This implies that there are no nodes of H that have less than the average number of blocks. Thus, at any given node of H, we must have exactly $n(G)$ G-blocks.

(iii) By definition, $\chi(H)$ counts the minimum number of independent sets. Thus, we must show that all the independent color classes are of the same order. Suppose that the order of the ith color class is k_i. Further, we may assume without loss of generality that:

$$k_1 \leq k_2 \leq \cdots \leq k_{\chi(H)} = k$$

Suppose to the contrary that not all color classes are the same size. Then there exists a j such that $k_j = k$ and $k_i < k$ for all $i < j$. Note that:

$$\sum_{i=1}^{\chi(H)} k_i = n(H). \tag{4.1}$$

Further note that:

$$\begin{aligned}\sum_{i=1}^{\chi(H)} k_i &= \sum_{i=1}^{j-1} k_i + \sum_{i=j}^{\chi(H)} k_i \\ &= \sum_{i=1}^{j-1} k_i + (\chi(H) - j + 1)k \\ &< \sum_{i=1}^{j-1} k + (\chi(H) - j + 1)k = (j-1)k + (\chi(H) - j + 1)k \\ &= \chi(H)k.\end{aligned}$$

Applying this to Equation 4.1 yields:

$$n(H) < \chi(H)k \Rightarrow \frac{n(H)}{\chi(H)} < k.$$

By hypothesis, we must have that:

$$k > \frac{n(H)}{\chi(H)} = \alpha(H). \tag{4.2}$$

However, the size of the largest color class must be no larger than the independence number. Thus, we must have that $k \leq \alpha(H)$. Applying this to Equation 4.2, we have that $\alpha(H) < k \leq \alpha(H)$, a contradiction. Thus, each color class must be of the same size k. Applying this to Equation 4.1 shows that $\chi(H)k = n(H)$ or equivalently $k = \frac{n(H)}{\chi(H)} = \alpha(H)$.

(iv) Assume that G is not the disjoint union of P_2's. Let $u \in V(H)$ be such that $deg_H(u) = \delta(H)$. If $\delta(H) = n(G)$, then there is a G-block, $A \in \mathscr{D}$, represented by u in $I(\mathscr{D})$ that intersects exactly $n(G)$ other blocks. Let $\mathscr{B}$ be the set of G-blocks that share a common node with A. For each node v of A we must choose $n(G) - 1$ distinct blocks to intersect A at v. Since $|\mathscr{B}| = n(G)$, we have exactly $n(G)$ choices for each node of A, where each of these choices leave out a distinct element of $\mathscr{B}$. If we use $n(G)$ different choices, then every pair of elements of $\mathscr{B}$ share a common node with A. This would result in a clique of size $n(G) + 1 > \omega(H)$, which is a contradiction. Thus, we can make at most $n(G) - 1$ different choices to place on each node of A. Note that each choice leaves out exactly one distinct block of $\mathscr{B}$. Thus, A shares the same node set with at least $n(G) - 2$ other blocks. Hence, we have $n(G) - 1$ copies of G that share the same node set. Since our decomposition is an edge partition, it follows that no two blocks share a common edge. Since H is simple, it follows that the most edges that this set of shared nodes can have is:

$$\frac{n(G)(n(G)-1)}{2}.$$

Thus, it follows that:

$$(n(G)-1)e(G) \leq \frac{n(G)(n(G)-1)}{2}$$

$$\Rightarrow e(G) \leq \frac{n(G)}{2}.$$

However, this implies that G either has an isolated node or that G is the disjoint union of P_2's. In either case, we have a contradiction. Thus $\delta(H) \geq n(G) + 1$.

(v) Note that by (iii), H has $\chi(H)$ independent sets of size $\alpha(H)$. Each of these independent sets correspond to a clique of size $\alpha(H)$ in $\overline{H}$. Each of these cliques contain $\frac{\alpha(H)(\alpha(H)-1)}{2}$ edges and there are $\chi(H)$ of these cliques. Thus we have:

$$2e(\overline{H}) \geq \chi(H)\alpha(H)(\alpha(H)-1)$$

$$= n(H)(\alpha(H) - 1).$$

Note that:

$$2e(H) = n(H)(n(H) - 1) - 2e(\overline{H})$$

$$\leq n(H)(n(H) - 1) - n(H)(\alpha(H) - 1) = n(H)(n(H) - \alpha(H)).$$

(vi) By Theorem 3.2.1, we have $\overline{d}(H) = 2e(G)$. However, $2e(G) \leq \Delta(G)n(G)$ by Proposition 1.1.2. Since G is a fully automorphic divisor of H, we have that $n(G) = \chi(H)$. Further, we have $\Delta(G) < \delta(H)$ by (iv). Thus, $2e(G) < \delta(H)\chi(H)$. Hence, by transitivity, we have $\overline{d}(H) < \delta(H)\chi(H)$.

(vii) Note that $n(H)e(G) = e(H)$ by Theorem 3.2.1. From (v), we have $2e(H) \leq n(H)(n(H)-\alpha(H))$. Thus we have:

$$2n(H)e(G) \leq n(H)(n(H) - \alpha(H))$$

$$\Rightarrow 2e(G) \leq n(H) - \alpha(H).$$

Since $n(H) = \alpha(H)\chi(H)$ by (i), we have that:

$$n(H) - \alpha(H) = \alpha(H)(\chi(H) - 1).$$

■

Note that by Theorem 4.2.2, any fully automorphic decomposition is $n(G)$-balanced. Further, since $n(G)$ G-blocks meet at any node of H, these blocks form a local clique of size $n(G) = \omega(H)$. However, we may also have non-local cliques. In terms of a design, we already have that the design is balanced and symmetric. Further, each of the independent sets of H are of the same order. Therefore, the independent sets of $I(\mathscr{D})$ have the same order. This implies that a fully automorphic decomposition is an example of a resolvable design.

Theorem 4.2.3 *Suppose that H admits a fully automorphic simple G-decomposition. Then H is $n(G)(n(G) - 1)$-regular and $G \cong K_{n(G)}$.*

Proof.

Let $\mathscr{D}$ be a fully automorphic simple G-decomposition of H. In a fully automorphic G-decomposition, exactly $n(G)$ G-blocks meet at any node of H by Theorem 4.2.2. Thus, for any G-block $B \in \mathscr{D}$, B contains exactly $n(G)$ distinct nodes of H. Further, each of these nodes appear in exactly $n(G) - 1$ other G-blocks. Since $\mathscr{D}$ is simple, this is exactly the number of G-blocks that B is adjacent to. Thus, B is adjacent to $n(G)(n(G) - 1)$ other blocks. Ergo, if $v_B \in V(I(\mathscr{D}))$ represents the block B, we must have that:

$$deg_{I(\mathscr{D})}(v_B) = n(G)(n(G) - 1).$$

Since B was chosen arbitrarily, it follows that for any $v \in I(\mathscr{D})$, we have that:

$$deg_{I(\mathscr{D})}(v) = n(G)(n(G) - 1).$$

This implies that $I(\mathscr{D})$ is $n(G)(n(G) - 1)$-regular. Since $\mathscr{D}$ is automorphic, it follows that H is $n(G)(n(G) - 1)$-regular. From Theorem 3.2.1, we have that:

$$\overline{d}(H) = 2e(G) = n(G)(n(G) - 1).$$

Thus $e(G) = \frac{n(G)(n(G)-1)}{2}$, which implies that G is complete. ∎

We note that if G is complete, then any decomposition of a simple graph, fully automorphic or otherwise, is simple by Proposition 2.3.6.

In Theorem 4.2.3, we gave a regularity result for a certain class of fully automorphic G-decompositions. In the following theorem, we expand on these results.

Theorem 4.2.4 *Suppose that H admits a fully automorphic G-decomposition.*

(i) If G is a d-regular graph, then H must be $n(G)d$-regular.

(ii) If G is not a disjoint union of P_2's and the smallest elements of $dseq(G)$ are 1 and a, where $a \geq 2e(G) - n(G) + 1$, then H must be $2e(G)$-regular.

(iii) If $G \cong P_4$, then H is 6-regular.

Proof.

(i) Since H admits a fully automorphic decomposition with respect to G, it follows from Theorem 4.2.2 that exactly $n(G)$ G-blocks meet at every node of H. Since G is d-regular, each of these blocks contribute exactly d edges. Thus, each node of H is incident with exactly $n(G)d$ edges. Hence, H is $n(G)d$-regular by definition.

(ii) Since G is not a disjoint union of P_2's and H admits a fully automorphic G-decomposition, it follows from Theorem 4.2.2 that $\delta(H) \geq n(G)+1$. Theorem 4.2.2 also implies that exactly $n(G)$ G-blocks meet at any given node of H. Since the smallest elements of $dseq(G)$ are 1 and a, it follows that the smallest admissible degrees for nodes of H are $n(G)$, $n(G) - 1 + a$, $n(G) - 2 + 2a$, etc. Since $\delta(H) \geq n(G) + 2$, it follows that $\delta(H) \geq n(G) - 1 + a$. However, $a \geq 2e(G) - n(G) + 1$ by definition. Thus $\delta(H) \geq 2e(G)$. But we have $\overline{d}(H) = 2e(G)$ from Theorem 3.2.1. Since H has no nodes of degree lower than $2e(G)$, it follows that H has no nodes of degree higher than $2e(G)$ either. Hence, H is $2e(G)$-regular.

(iii) Since $G \cong P_4$ is not a disjoint union of P_2's and H admits a fully automorphic G-decomposition, it follows from Theorem 4.2.2 that $\delta(H) \geq n(G) + 1 = 5$. Since $\overline{d}(H) = 2e(G) = 6$ by Theorem 3.2.1, it follows that we need only show that H has no nodes of degree five. Suppose to the contrary that H has a node of degree five. This means that there is a G-block A that intersects exactly five others: B_1, B_2, B_3, B_4, and B_5. At every node of A, exactly three of these blocks must intersect A. For convenience of exhibition, we denote these blocks by their indices. The possible combinations of blocks that can intersect at a node of A are 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345. At most, we can have a clique of order four, and one of these blocks is A. Thus, at most three of the B_i mutually intersect. Further, since at most four of these combinations are used, at least two pairs of numbers never appear on a node of A. Without loss of generality, suppose that the pairs 12 and 13 never appear on a node of A, this leaves 145, 234, 235, 245, and 345. We cannot allow any two of 234, 235, and 245 as this would result in a clique of size $n(G)+1$. Thus, we have 145 and 345 as well as one of the above blocks. In any case, B_5 shares a common node set with A. Since $G = P_4$ is self-complementary, it follows that any other P_4-block may share at most one node with A (otherwise that block will share at least one edge with either A or B_5). However, B_4 shares two nodes with A, a contradiction. Thus, H has no nodes of degree five. Since H has no nodes of degree lower than six, it may have no nodes of higher degree either. Hence, H must be regular of degree six. ■

Note that we proved a similar statement about graphs that admit an automorphic decomposition with respect to a regular graph G in Theorem 4.1.1. That theorem used the hypothesis that the decomposition was simple and the proof was considerably longer. This suggests that the assumption of a fully automorphic decomposition is a much stronger condition than that of a simple automorphic decomposition.

Proposition 4.2.5 *Complete graphs cannot host fully automorphic decompositions.*

Proof.

Note that complete graphs of even order do not have automorphic decompositions by Corollary 3.2.3. Hence, it follows that we need only consider complete graphs of odd order, say K_{2n+1}. Observe that $\chi(K_{2n+1}) = 2n+1$. Thus, we need to find a prototype G such that $n(G) = 2n+1$. By Theorem 3.2.1, for a G-decomposition of K_{2n+1} to be automorphic, we must have:

$$e(G) = \frac{e(K_{2n+1})}{n(K_{2n+1})} = n.$$

Since each edge covers at most two distinct nodes, it follows from the Proposition 1.1.6 that G must contain at least one isolated node. However, since we do not allow prototypes with isolated nodes, it follows that K_{2n+1} has no fully automorphic decomposition. ■

4.3 Automorphic Hosts with a Pendant Node

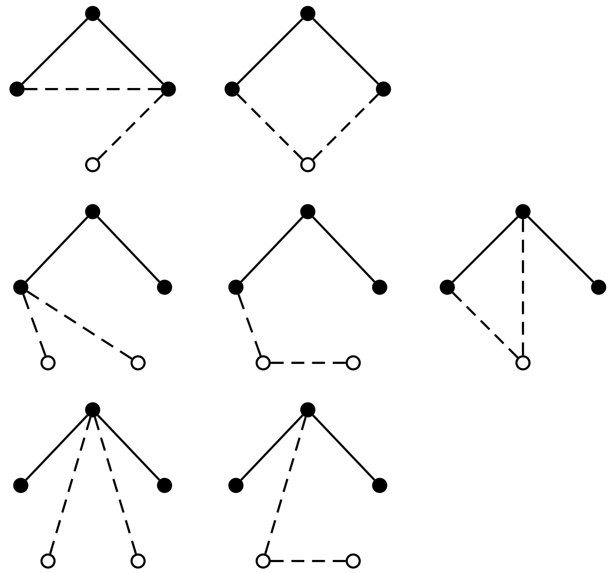
We have conjectured that only regular graphs of even degree will admit an automorphic decomposition. Certainly, the author has found it difficult to produce a counterexample. While some progress has been made in special cases of this conjecture, unfortunately very little is known at this time. In particular, it is unknown if an automorphic host can have a pendant node. In this section, we present an informal discussion of this problem. Several ideas presented in this section are fairly intuitive and would become unnecessarily complicated by a formal treatment. We will, however, formalize certain ideas when appropriate. While the discussion is focused on hosts with pendant nodes, we expect that many ideas presented here will generalize to hosts with nodes of degree $d < 2e(G)$. Assuming that H admits an automorphic G-decomposition, we know the following about H:

- If $\chi(H) \leq 3$, then H is $2e(G)$-regular.
- If $\chi(H) = n(G)$ and G is d-regular, then H is $n(G)d$-regular.
- If $\chi(H) = n(G)$, G is not a disjoint union of P_2's, and the smallest elements of $dseq(G)$ are 1 and a, where $a \geq 2e(G) - n(G) + 1$, then H must be $2e(G)$-regular.
- If $G \cong P_4$ and $\chi(H) = 4$, then H is 6-regular.
- If the decomposition is simple and G is d-regular, then H is $n(G)d$-regular.

Note that if $\delta(H) = 1$ and H admits an automorphic G-decomposition $\mathscr{D}$ then there must be a G-block $G_1 \in \mathscr{D}$ which intersects exactly one other G-block G_2. In general, if H has a node of degree k, then there is a G-block that intersects exactly k others. This places some restriction on the structure of H.

Proposition 4.3.1 *Let H be a graph with no connected component isomorphic to P_2 and $\exists v \in V(H)$ such that $deg_H(v) = 1$. Let $\mathscr{D}$ be an automorphic G-decomposition of H, where $G_1 \in \mathscr{D}$ is the G-block representing v in $I(\mathscr{D})$. Then there is a unique G-block, G_2, intersecting G_1. Further, G_1 and G_2 share at most $n(G) - 1$ common nodes in H.*

Figure 4.1 Possible P_3 Intersections Representing a Node of Degree One



Proof.

If G_1 intersects more than one other block, then the vertex in $I(\mathscr{D})$ that represents G_1 would have degree greater than one. This contradicts G_1 representing a node of degree one. Let G_2 be the unique G-block that shares a node with G_1. If G_1 and G_2 share $n(G)$ common nodes in H, then any block that intersects G_2 would also intersect G_1, contrary to G_1 representing a pendant node. Thus, G_1 and G_2 may only intersect each other. If G_1 and G_2 share $n(G)$ common nodes, any G-block intersecting G_2 would also intersect G_1. This contradicts G_1 representing a node of degree one. Thus, the vertices representing these blocks in the G-decomposition graph would be in a connected component with no other vertices. This connected component would be isomorphic to P_2. Since $\mathscr{D}$ is an automorphic decomposition, it follows that H would have a connected component isomorphic to P_2, a contradiction. ■

It is of interest how the block G_1 may intersect G_2. As a motivating example, we give all possible intersections where $G \cong P_3$ in Figure 4.1. In Figure 4.1, the solid nodes and solid edges represent G_1. The hollow nodes and dashed edges are unique to G_2. Solid nodes are shared by both G_1 and G_2. It is feasible to assume that as G becomes more complex, the number of possible intersections increases. By using simple counting arguments, we can give upper bounds for the number of possible intersections.

Theorem 4.3.2 *Suppose that H admits an automorphic G-decomposition $\mathscr{D}$. Suppose that G_1 is a G-block represented by a vertex of degree one in $I(\mathscr{D})$.*

(i) If $\mathscr{D}$ is λ-uniform, then the number of possible intersections is bounded above by:

$$\binom{n(G)}{\lambda}^2.$$

(ii) If $\mathscr{D}$ is λ-bounded, then the number of possible intersections is bounded above by:

$$\sum_{k=1}^{\lambda} \binom{n(G)}{k}^2.$$

(iii) For general $\mathscr{D}$, the number of possible intersections is bounded above by:

$$\sum_{k=1}^{n(G)} \binom{n(G)}{k}^2.$$

Proof.

(i) By Proposition 4.3.1, since G_1 represents a node of degree one, it intersects a unique block G_2. Since $\mathscr{D}$ is λ-uniform, these blocks share λ common nodes. Thus, we must choose λ of the $n(G)$ nodes of G_1 to intersect with G_2. There are $\binom{n(G)}{\lambda}$ ways of doing this. We must also choose λ of the $n(G)$ nodes of G_2 to intersect with G_1. There are $\binom{n(G)}{\lambda}$ ways of doing this. Thus, by the multiplicative principle, there are $\binom{n(G)}{\lambda}^2$ ways of choosing the nodes for the intersection. However, not all of these may be non-isomorphic. Thus, the number of possible intersections is bounded above by $\binom{n(G)}{\lambda}^2$.

(ii) Since $\mathscr{D}$ is λ-bounded, then G_1 and G_2 may share k common nodes where $1 \leq k \leq \lambda$. If G_1 and G_2 share k common nodes, then the number of possible intersections is bounded above by $\binom{n(G)}{k}^2$ by (i). Summing over all values of k, we see that the number of possible intersections for a λ-bounded automorphic decomposition is bounded above by:

$$\sum_{k=1}^{\lambda} \binom{n(G)}{k}^2.$$

(iii) A general G-decomposition is $n(G)$-bounded by definition. Thus applying (ii), the number of possible intersections is bounded above by:

$$\sum_{k=1}^{n(G)} \binom{n(G)}{k}^2.$$

■

It should be noted that these bounds can be improved significantly if we put additional restrictions on the structure of G or H. For example, if we assume that H has no connected component isomorphic to P_2 then the bound given in Theorem 4.3.2(iii) can be improved to

$$\sum_{k=1}^{n(G)-1} \binom{n(G)}{k}^2$$

by applying Proposition 4.3.1. Restrictions on G will also improve these bounds significantly. For example if $G \cong P_3$, then it does not matter which of the two endpoints that we choose. Thus, the bounds would be improved by a factor of one fourth.

While it may be possible to use similar methods to give bounds on the number of intersections on a block representing a node of arbitrary degree, these bounds are much more complicated, and not nearly as tight. However, we have included these bounds for completeness.

Theorem 4.3.3 *Suppose that H admits an automorphic G-decomposition $\mathscr{D}$. Suppose that G_0 is a G-block representing a vertex of degree k in $I(\mathscr{D})$.*

(i) If $\mathscr{D}$ is λ-uniform, then the number of possible intersections is bounded above by:

$$\binom{n(G)}{\lambda}^{2k}\left[\binom{n(G)}{\lambda}+1\right]^{\frac{k(k-1)}{2}}.$$

(ii) If $\mathscr{D}$ is λ-bounded, then the number of possible intersections is bounded above by:

$$\left[\sum_{\ell=1}^{\lambda}\binom{n(G)}{\ell}^2\right]^k\left[\sum_{\ell=0}^{\lambda}\binom{n(G)}{\ell}^2\right]^{\frac{k(k-1)}{2}}.$$

(iii) For general $\mathscr{D}$, the number of possible intersections is bounded above by:

$$\left[\sum_{\ell=1}^{n(G)}\binom{n(G)}{\ell}^2\right]^k\left[\sum_{\ell=0}^{n(G)}\binom{n(G)}{\ell}^2\right]^{\frac{k(k-1)}{2}}.$$

Proof.

(i) Since G_0 represents a node of degree k, it must intersect exactly k blocks in H. Suppose that these blocks are $G_1, ..., G_k$. As shown in Theorem 4.3.2, there are at most $\binom{n(G)}{\lambda}^2$ ways that G_0 can intersect with G_i. Since this holds true for all i, there are at most $\binom{n(G)}{\lambda}^{2k}$ intersections, assuming that G_i and G_j do not intersect for $i, j > 0$. However, we must consider these possible intersections as well. There is exactly one way that G_i and G_j do not intersect. By Theorem 4.3.2, there are at most $\binom{n(G)}{\lambda}^2$ ways that they do intersect. Thus, there are at most $\binom{n(G)}{\lambda}^2+1$ possible intersections of G_i and G_j, including the empty intersection. Hence, for all unordered pairs of G_i and G_j, we have at most $[\binom{n(G)}{\lambda}^2+1]^{\frac{k(k-1)}{2}}$ intersections of these pairs. Thus, by the multiplication principle, we have at most

$$\binom{n(G)}{\lambda}^{2k}\left[\binom{n(G)}{\lambda}+1\right]^{\frac{k(k-1)}{2}}$$

intersections involving G_0.

(ii) By Theorem 4.3.2, there are at most $\sum_{\ell=1}^{\lambda}\binom{n(G)}{\ell}^2$ ways for G_0 to intersect with G_i. Thus, assuming that G_i and G_j do not intersect, the number of possible intersections is at most:

$$\left[\sum_{\ell=1}^{\lambda}\binom{n(G)}{\ell}^2\right]^k.$$

Note that there are at most $\binom{n(G)}{\ell}^2$ ways for G_i and G_j to share ℓ nodes by Theorem 4.3.2. Hence, there are at most $\sum_{\ell=0}^{\lambda}\binom{n(G)}{\ell}^2$ ways for G_i and G_j to intersect. Over all possible pairs of G_i and G_j, the number of intersections is at most:

$$\left[\sum_{\ell=0}^{\lambda}\binom{n(G)}{\ell}^2\right]^{\frac{k(k-1)}{2}}.$$

Thus by the multiplication principle, the number of intersections involving G_0 is at most:

$$\left[\sum_{\ell=1}^{\lambda}\binom{n(G)}{\ell}^2\right]^k\left[\sum_{\ell=0}^{\lambda}\binom{n(G)}{\ell}^2\right]^{\frac{k(k-1)}{2}}.$$

(iii) A general G-decomposition is $n(G)$-bounded by definition. Thus applying (ii), the number of possible intersections is bounded above by:

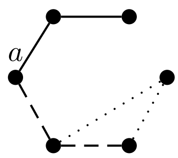
$$\left[\sum_{\ell=1}^{n(G)}\binom{n(G)}{\ell}^2\right]^k\left[\sum_{\ell=0}^{n(G)}\binom{n(G)}{\ell}^2\right]^{\frac{k(k-1)}{2}}.$$

■

While the prototype used in Figure 4.1 is simple, it brings up several interesting points. First of all, note the organization of Figure 4.1. The intersections of the first row leave no pendant nodes of degree one *exposed*, while the intersections in the third row leave two pendant nodes of degree one exposed. Note that these intersections represent a *single* vertex of degree one. Thus, if our intersection is from the third row, it must be balanced by an intersection from the first row. However, none of the intersections in the first row originate from simple decompositions. Thus, if we restrict our attention to simple decompositions, we would only allow the first two entries in the second row. This gives rise to a general principle.

Proposition 4.3.4 *Let H be a graph such that $\delta(H) = 1$. If $\mathscr{D}$ is a λ-bounded automorphic G-decomposition of H, then G may have at most $1 + \lambda$ pendant nodes.*

Figure 4.2 An Exposed Node



Proof.

Suppose that G has ℓ pendant nodes. Let k be the number of pendant nodes in H. Since $\delta(H) = 1$, Proposition 4.3.1 implies that there is a G-block $G_1 \in \mathscr{D}$ that intersects exactly one other G-block, G_2. Since $\mathscr{D}$ is a λ-bounded decomposition, G_1 and G_2 share at most λ common nodes in H. Thus, at least $\ell - \lambda$ pendant nodes of G_1 are exposed in this intersection. Since this holds for each of the k G-blocks representing a node of degree one, we have that H has at least $k(\ell - \lambda)$ pendant nodes. However, H is assumed to have only k pendant nodes, thus $k \geq k(\ell - \lambda)$. Solving for ℓ yields $\ell \leq 1 + \lambda$. ■

We also mentioned that the types of intersections must balance each other out. In other words, if we have an intersection with two exposed pendant nodes, we must have an intersection with no pendant nodes to balance it out. However, the converse is not true. If we have an intersection with no exposed pendant vertices, it may be balanced by an intersection representing a vertex of higher degree that has an exposed pendant node in the intersection. Thus, we have at least as many intersections of the first type as intersections of the third type. Further, since the intersections that represent a vertex of degree one in $I(\mathscr{D})$ may introduce nodes of higher degree in H (and vice versa), any system that defines this relationship would be quite complicated. As shown in Figure 4.2, the node labelled a is left exposed in a block of degree two and a block of degree three. Scenarios such as this, cause great difficulties in counting arguments. Even the notation for the variables used in the linear diophantine equations would be necessarily complex. Previously, we have used linear diophantine equations in several of our necessary conditions. As a reminder these were:

(i) Theorem 2.3.7 - For each $i \in \{1, ..., n(H)\}$ there exist non-negative integers $x_{i,1}, ..., x_{i,n(G)}$ such that:

$$\sum_{j=1}^{n(G)} x_{i,j} deg_G(u_j) = deg_H(v_i).$$

(ii) Theorem 3.2.1 - There exist non-negative integers $d_{\delta G}, ..., d_{\omega(H)\Delta(G)}$ such that:

$$\sum_{i=\delta(G)}^{\omega(H)\Delta(G)} d_i = n(H) \quad \text{and} \quad \sum_{i=\delta(G)}^{\omega(H)\Delta(G)} id_i = 2e(G)n(H).$$

(iii) Theorem 3.2.1 - There exist non-negative integers $b_1, ..., b_{\omega(H)}$ such that:

$$\sum_{i=1}^{\omega(H)} b_i = n(H) \quad \text{and} \quad \sum_{i=1}^{\omega(H)} ib_i = n(G)n(H).$$

With these in mind, we present the following theorem.

Theorem 4.3.5 *Let H and G be graphs. Let n_i be the number of nodes in H of degree i. If H admits an automorphic G-decomposition, then there must be a set of non-negative integers $t(a_{i,j,1}, ..., a_{i,j,\Delta(H)})$ such that for $i = 1, ..., \Delta(H)$:*

$$\sum_{j=1}^{n_i} t(a_{i,j,1}, ..., a_{i,j,\Delta(H)}) = n_i \quad \textit{and}$$

$$\sum_{i,j \geq 1} a_{i,j,k} t(a_{i,j,1}, ..., a_{i,j,\Delta(H)}) \geq n_k.$$

Proof.

Suppose that $\mathscr{D}$ is an automorphic G-decomposition of H. Let $t(a_{i,j,1}, ..., a_{i,j,\Delta(H)})$ be the number of G-blocks in $\mathscr{D}$ that are represented by a vertex of degree i in $I(\mathscr{D})$ with $a_{i,j,k}$ nodes of degree k exposed in H, where $j = 1, ..., n_i$. Since n_i is the number of nodes in H with degree i, it follows that for all i we have that:

$$\sum_{j=1}^{n_i} t(a_{i,j,1}, ..., a_{i,j,\Delta(H)}) = n_i.$$

Further, since the number of exposed vertices of degree k is overestimated by:

$$\sum_{i,j\geq 1} a_{i,j,k}t(a_{i,j,1},...,a_{i,j,\Delta(H)}),$$

it follows that:

$$\sum_{i,j\geq 1} a_{i,j,k}t(a_{i,j,1},...,a_{i,j,\Delta(H)}) \geq n_k.$$

■

It should be noted that the resulting system has more variables than constraining equations. As such, it is likely that there are many solutions to the problem. It seems to be the case that all of the linear diophantine systems that appear in the study of automorphic decompositions are underdetermined. While these systems are interrelated, it may be difficult to get information from them.

Frequently, we have used properties of H to derive properties and necessary conditions for an automorphic decomposition. In particular, suppose $\mathscr{D}$ is an automorphic decomposition and $\mathcal{S}_0$ is some structure in H. For instance, $\mathcal{S}_0$ could be a pendant node, an induced subgraph, or an independent set, etc. Since $\mathscr{D}$ is automorphic, $\mathcal{S}_0$ must also occur as the same kind of structure in $I(\mathscr{D})$. Thus, we must have a set of G-blocks in H whose intersection pattern forms $\mathcal{S}_0$. We can think of these blocks as a *first order structure* $\mathcal{S}_1$ derived from $\mathcal{S}_0$. Note that $\mathcal{S}_1$ need not be unique. Many of the necessary conditions of Theorem 3.2.1 were proven using an elementary form of this idea.

In an automorphic decomposition, any first order intersection $\mathcal{S}_1$ must be represented as an intersection of G-blocks (not necessarily disjoint from the first set). Thus, we may define a new set of intersections that represent $\mathcal{S}_1$, which we may describe as a *second order structure* $\mathcal{S}_2$ derived from $\mathcal{S}_0$ and $\mathcal{S}_1$. Continuing in this way, we can describe a *k-th order structure* $\mathcal{S}_k$ for arbitrary k. Note that as k increases without bound, H will eventually appear as some $\mathcal{S}_k$.

Intuitively, these intersections will become more complicated and more numerous as k increases. However, these intersections may give useful information about automorphic decompositions. For instance, the first graph in Figure 4.1 has a node of degree four. Since this is greater than the order of P_3, it follows from Theorem 2.4.5 that the two hollow vertices must be adjacent. Hence, we have imposed additional structure on H. Continuing onward with higher order intersections should impose additional structure which may eliminate certain possibilities.

CHAPTER 5
AUTOMORPHIC DIVISORS

We have conjectured that every graph is an automorphic divisor. There are several facts that lead credence to this conjecture. The first is that any graph with a ρ-valuation will be an automorphic divisor of the complete graph [66]. There are no known examples of graphs that do not have a ρ-valuation [31]. Second, any graph is a cyclic divisor of a circulant and the resulting G-decomposition graph will be a circulant (see Theorem 2.3.10, Corollary 2.3.12, and Theorem 2.4.7).

While it seems plausible that such a conjecture would be true, proving it may be much more difficult. We suspect that proving this conjecture is roughly equivalent in difficulty to proving Ringel's Conjecture. In this chapter, we will extend the classes of graphs that are automorphic divisors. We will also examine the problem of specialized divisors introduced in previous chapters. In particular, we will look at persistent automorphic divisors and fully automorphic divisors.

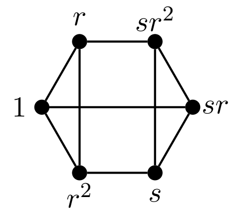
5.1 Transulants, Translational Decompositions, and Γ-valuations

Definition 5.1.1 A generalized definition of the circulant - *Let* Γ *be a group of order* n *and let* S *be a set whose elements are of the form* xy^{-1}. *The* Γ-transulant, *denoted* $C_\Gamma(S)$, *is the undirected graph whose vertices are elements of* Γ *and whose edge set is:*

$$E = \{xy : xy^{-1} \in S \quad or \quad yx^{-1} \in S\}.$$

Remark 5.1.2 Γ*-transulants should not be confused with Cayley graphs which are also generated by groups. First, Cayley graphs are directed, while* Γ*-transulants are not. Second, the difference set of a Cayley graph is always a generating set of the group. While,* S *being a generating set of* Γ *is necessary for the connectivity of* $C_\Gamma(S)$*, we do not require this assumption. Further, while both the Cayley graph and* $C_\Gamma(S)$ *allow for involutions in the difference set, we*

Figure 5.1 Γ-Transulant $C_{D_6}(r, sr)$



tend to avoid involutions due to the complications that they create. Finally, Cayley graphs are used as a means of studying the underlying group, while we are primarily interested in the graphs themselves [75].

Example 5.1.3 *Let* $\Gamma = D_6 = \{1, r, r^2, s, sr, sr^2\}$ *where* $r^3 = 1$*,* $s^2 = 1$ *and* $rs = sr^2$*. Take* $S = \{r, sr\}$*. The associated* Γ*-transulant is given in Figure 5.1.*

Note that in the special case where $\Gamma = \mathbb{Z}_n$, we will still denote the circulant as $C_n(S)$ rather than $C_{\mathbb{Z}_n}(S)$.

Definition 5.1.4 A generalization of cyclic decompositions - *Let* Γ *be a group and let* H *and* G *be graphs. A* Γ-translational G-decomposition $\mathscr{D}$ *of a graph* H *is one in which given any two* G*-blocks of* $\mathscr{D}$*, say* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *there is an element* $g \in \Gamma$ *such that* $\mathcal{B}_1 g = \mathcal{B}_2$*.*

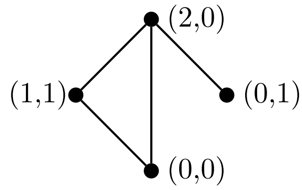
Definition 5.1.5 A generalization of $\mathbb{Z}_n$-valuations - *Let* Γ *be a group and let* G *be a graph. A* Γ-labelling of G *is an injective map* $f : V(G) \to \Gamma$*. Any* Γ*-labelling* f *of* G *induces an edge labelling* f^* *on* $E(G)$ *by* $f^*(xy) = xy^{-1}$ *for all edges* $xy \in E(G)$*. In this case,* xy *will be an edge of length* xy^{-1}*. The set of edge labels induced by* f *is denoted:*

$$f^*(E(G)) = \{xy^{-1}, yx^{-1} : xy \in E(G)\}.$$

Similarly, this concept can be extended to all pairs of vertices with:

$$f^*(G) = \{xy^{-1}, yx^{-1} : x, y \in V(G), x \neq y\}.$$

Figure 5.2 A $(\mathbb{Z}_3 \times \mathbb{Z}_3)$-Valuation of a Graph



Note that $f^*(E) \subseteq f^*(G)$ for all f and G. A Γ-labelling f of G is a Γ-valuation *if and only if the induced edge labelling f^* is injective and none of the edge labels are involutions in Γ. If f is a Γ-valuation on G such that $f^*(G) \subseteq f^*(E(G))$, then we say f is a* closed Γ-valuation *of G. If f is a Γ-valuation on G such that every non-involutory element of Γ is either an edge label of G or its inverse is an edge label of G, then we say that f is a* complete Γ-valuation *of G.*

An example of a $\mathbb{Z}_3 \times \mathbb{Z}_3$-valuation is given in Figure 5.2.

The following theorem generalizes Theorem 2.3.10, Theorem 2.3.14, and Theorem 2.4.7 with the use of translational decompositions, transulants, and Γ-valuations.

Theorem 5.1.6 *Let G, H be graphs, Γ be a group, and let $S \subseteq \Gamma$.*

(i) A Γ-translational G-decomposition of $H = C_\Gamma(S)$ exists if and only if there is a Γ-valuation of G, f, such that $f^*(E(G)) = S$.

(ii) If H admits a Γ-translational G-decomposition, then H is a Γ-transulant.

(iii) Let f be a Γ-valuation of G. Define $S = f^*(E(G))$ and $S' = f^*(G)$. If $\mathscr{D}$ is the Γ-translational decomposition of $H = C_\Gamma(S)$ induced by f, then $I(\mathscr{D}) \cong C_\Gamma(S')$.

Proof.

(i) Let $\mathscr{D}$ be a Γ-translational G-decomposition of H. Let G_1 be a G-block in the decomposition. We must show that the edges of G_1 have distinct lengths in H.

Suppose that G_1 has two edges of the same length. Let $(x, ax), (y, ay) \in E(G_1)$ where $a \in S$ and $x \neq y$. By definition of Γ-translational decomposition, if G_2 is any other G-block in the decomposition, there exists a $g \in \Gamma$ such that $G_2 = G_1 g$. In particular, this holds if $g = x^{-1}y$. Since $(x, ax) \in E(G_1)$, it follows that $(xg, axg) \in E(G_2)$. By definition, we have $(xg, axg) = (y, ay) \in G_1$. Thus, the edge (y, ay) is in two distinct G-blocks, contrary to $\mathscr{D}$ being an edge decomposition. Thus, the edge lengths on G_1 must be unique. As such we have a Γ-valuation of G.

Conversely, let f be a Γ-valuation of G. For $v_i \in V(G)$, let $f(v_i) = f_i$. Denote the nodes of H in such a way that $v_i = v_{f_i}$. In other words, each node of G is identified with a unique node of H. For $v_i v_j \in E(G)$, let $d_{ij} = f^*(v_i v_j)$ be the induced edge label of $v_i v_j$. Since each node of G is identified with a unique node of H, it follows that d_{ij} is an edge length in H. This implies that the edges of G have in H mutually distinct lengths. This implies the existence of a Γ-translational decomposition.

(ii) Suppose that H admits a Γ-translational G-decomposition, $\mathscr{D}$. Let $G_e \in \mathscr{D}$ and note that for all G-blocks, $G_g \in \mathscr{D}$, there exists $g \in \Gamma$ such that $G_e g = G_g$. If $v_0 \in V(G_e)$, then by definition of a Γ-translational decomposition, $v_0 g \in V(G_g)$. This implies that $v_0 g \in V(H)$ for all $g \in \Gamma$. As v_0 was chosen arbitrarily, this implies that $V(H) = \Gamma$. Let $\pi_g : v_0 \to v_0 g = v_g$. Note that this gives a natural identification with Γ given by:

$$\pi_h(g) \equiv \pi_h(v_g) = \pi_h(\pi_g(v_0)) = \pi_{gh}(g_0) = v_0 gh = v_{gh} \equiv gh.$$

As such, the blocks may be identified as $G_g = G_e g$ for all $g \in \Gamma$. Note that $G_g = \pi_g[G_e]$. Define

$$S = \{xy^{-1}, yx^{-1} : xy \in E(G_e)\}.$$

We claim that $H \cong C_\Gamma(S)$. Suppose that $ab \in E(H)$. Since $\mathscr{D}$ is a G-decomposition, then for some $G_g \in \mathscr{D}$, $ab \in E(G_g)$. Since $\mathscr{D}$ is a Γ-translational decomposition, there is $xy \in E(G_e)$ such that $\pi_g(xy) = ab$. Without loss of generality, we may assume that $\pi_g(x) = a$ and $\pi_g(y) = b$. Thus $a = xg$ and $b = yg$. Since $xy^{-1}, yx^{-1} \in S$ by definition and $ab^{-1} = (xg)(g^{-1}y^{-1}) = xy^{-1}$, it follows that $ab \in E(C_\Gamma(S))$. Thus $E(H) \subseteq E(C_\Gamma(S))$.

Conversely, suppose that $cd \in E(C_\Gamma(S))$. This implies that $cd^{-1} = s$ for some $s \in S$. Let $x, y \in V(G_e)$ be such that $xy^{-1} = s$. From this it follows that $xy^{-1} = cd^{-1}$ or equivalently, $y^{-1}d = x^{-1}c = g$. Thus, we have that $c = xg$ and $d = xg$. Hence $cd \in E(G_g)$, and therefore $cd \in E(H)$. A similar argument holds for the case where $xy^{-1} = dc^{-1}$. Thus, $E(C_\Gamma(S)) \subseteq E(H)$. Ergo, $H \cong C_\Gamma(S)$.

(iii) Let f be a Γ-valuation and take $\mathscr{D}$ to be the Γ-translational decomposition induced by f. Suppose that $B \in \mathscr{D}$ is a G-block in the decomposition with array representation:

$$f(V(B)) =< f_1, f_2, ..., f_p > .$$

Since $\mathscr{D}$ is a Γ-translational decomposition, if A is any other G-block, then $A = Bg$ for some $g \in \Gamma$. Note that B is adjacent to blocks of the form $Bf_j^{-1}f_i$ as these blocks have f_i in the jth position. Similarly, by reversing the roles of i and j, we obtain that B is adjacent to $Bf_i^{-1}f_j$. As this covers all nodes in B, it follows that blocks of the above forms are the only ones with which B is adjacent. We define a mapping μ from the elements of $\mathscr{D}$ to the elements of Γ by $\mu : Bg \to g$. Since the decomposition is Γ-translational, this mapping is well defined. Thus $V(I(\mathscr{D})) = \Gamma$. Further, $xy \in E(I(\mathscr{D}))$ if and only if $xy^{-1} = f_i^{-1}f_j$ or $xy^{-1} = f_jf_i^{-1}$. Thus $I(\mathscr{D}) \cong C_\Gamma(S')$ where $S' = f^*(G)$.

■

In the future, we will use the convention that 1 is the identity element of a multiplicative group and that 0 is the identity of an additive group.

Note that arbitrary groups may have multiple non-identity involutions. Thus, it may be useful in future research to consider a labelling scheme that allows edges to be labelled with an involution. This may lead to an alternative algebraic construction. However, the intersection structure generated by such a decomposition would be much more difficult to predict. As such, we leave the study of these decompositions as a problem for future research.

Proposition 5.1.7 *Let G be a graph and let Γ be a group such that $i(\Gamma) = \{1\}$. If G has a complete Γ-valuation, then G is an automorphic divisor of a complete graph.*

Proof.

Since $i(\Gamma) = \{1\}$, it follows that the identity is the only element of Γ that is an involution. Suppose that G has a complete Γ-valuation, f. This implies that there exists a Γ-translational G-decomposition of $H = C_\Gamma(f^*(E(G)))$ by Theorem 5.1.6. By definition of complete Γ-valuation, every non-involutory element of Γ is an element of $f^*(E(G))$. By definition of Γ, this means that every non-identity element of Γ is an element of $f^*(E(G))$. Thus for all $x, y \in \Gamma$ such that $x \neq y$, we have that $xy^{-1} \in f^*(E(G))$. Thus any pair of nodes in H are adjacent. Hence, H is complete by definition. Ergo, this decomposition is automorphic by definition. ■

While it seems plausible that most, if not all, automorphic hosts are Γ-transulants, this is not the case. We showed that $H = \overline{K_3} \vee C_6(2,3)$ was a non-circulant automorphic host in Theorem 3.1.7. We now show that this graph is not isomorphic to any Γ-transulant.

Proposition 5.1.8 *$H = \overline{K_3} \vee C_6(2,3)$ is not isomorphic to $C_\Gamma(S)$ for any choice of S and Γ.*

Proof.

Since $n(H) = 9$, we must have that $o(\Gamma) = 9$. However, there are only two groups of order nine, namely $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ [26]. If $\Gamma = \mathbb{Z}_9$, then H is a circulant, which was shown to be impossible in Proposition 3.1.6. Thus, we need only show that $\Gamma \neq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Suppose to the contrary that $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ and that $H \cong C_\Gamma(S)$. Since H is regular of degree six, it follows that there are two non-involutory elements of Γ not contained in S. Further, these elements are inverses. Note that for all $a \in \Gamma$, $a + a + a = (0, 0)$, the identity. Thus, we must be able to partition the nodes of H into three independent sets. Note that 0 is not adjacent to 1 and 5. However, 1 and 5 are connected by an edge of length two. Since these were the only nodes not adjacent to 0, we have a contradiction. ∎

5.2 Subproducts and Subjoins of Graphs

In the following few sections, we look at methods for constructing new automorphic divisors from old ones. In particular, we are interested in knowing if G and H have Γ and Λ-valuations, respectively, will $G \boxtimes H$ or $G \vee H$ have a $(\Gamma \times \Lambda)$-valuation? We begin with an observation about Γ-labellings.

Proposition 5.2.1 *For $i = 1, ..., n$, let G_i be a graph with a Γ_i-labelling. If H is any graph with vertex set $V(G_1) \times \cdots \times V(G_n)$, then H has a $(\Gamma_1 \times \cdots \times \Gamma_n)$-labelling.*

Proof.

Let h_i be a Γ_i-labelling of G_i. Let h be a mapping from $V(G_1) \times \cdots \times V(G_n)$ to $\Gamma_1 \times \cdots \times \Gamma_n$ defined by:

$$h(v_1, ..., v_n) = (h_1(v_1), ..., h_n(v_n)).$$

Since all of the h_i are injective by hypothesis, it follows that h is as well. ∎

This proposition will assist us in creating the required valuations for both classes of graphs.

Proposition 5.2.2 *Let G and H be graphs and let Γ and Λ be groups. Suppose that G has a Γ-valuation, γ, and that H has a Λ-valuation, λ. Consider the graph $G \boxtimes H$ and the mapping $\mu : V(G) \times V(H) \to \Gamma \times \Lambda$, defined by $\mu(g, h) = (\gamma(g), \lambda(h))$. Let e be an edge of $G \boxtimes H$ such that e has a non-trivial label, i.e., $\mu^*(e) \neq (a, 1)$ or $\mu^*(e) \neq (1, b)$ for some $a \in \Gamma$ and $b \in \Lambda$. For all such e there is a unique $f \in E(G \boxtimes H)$ such that $e \neq f$ and $\mu^*(e) = \mu^*(f)$.*

Proof.

Let $s, t \in V(G)$ and let $u, v \in V(H)$ be such that $st \in E(G)$ and $uv \in E(H)$. By definition of strong product, $(s, u), (s, v), (t, u), (t, v) \in V(G \boxtimes H)$ and these vertices form a clique of order four in $G \boxtimes H$. The edge labels induced by μ are:

$$\mu^*((s,u),(s,v)) = (1, \lambda(u)\lambda(v)^{-1}),$$

$$\mu^*((t,u),(t,v)) = (1, \lambda(u)\lambda(v)^{-1}),$$

$$\mu^*((s,u),(t,u)) = (\gamma(s)\gamma(t)^{-1}, 1),$$

$$\mu^*((s,v),(t,v)) = (\gamma(s)\gamma(t)^{-1}, 1),$$

$$\mu^*((s,u),(t,v)) = (\gamma(s)\gamma(t)^{-1}, \lambda(u)\lambda(v)^{-1}),$$

$$\mu^*((s,v),(t,u)) = (\gamma(s)\gamma(t)^{-1}, \lambda(u)\lambda(v)^{-1}).$$

Notice that within the clique, each edge label is repeated exactly once. Thus, it remains to be shown that the non-trivial edge labels are not repeated outside of the clique. Suppose that there exist $wy \in E(G)$ and $xz \in E(H)$ such that:

$$\mu^*((w,x),(y,z)) = (\gamma(s)\gamma(t)^{-1}, \lambda(u)\lambda(v)^{-1}).$$

But, by definition of μ^*, we have that:

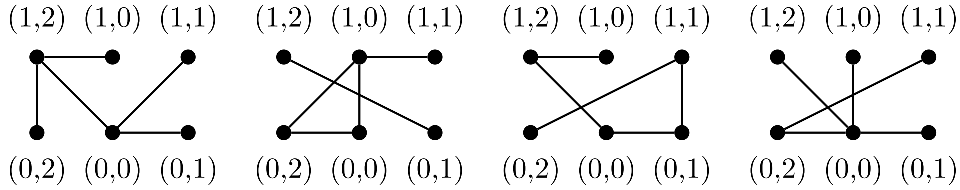
$$\mu^*((w,x),(y,z)) = (\gamma(w)\gamma(y)^{-1}, \lambda(x)\lambda(z)^{-1}).$$

This implies that $\gamma(w)\gamma(y)^{-1} = \gamma(s)\gamma(t)^{-1}$, contrary to γ being a Γ-valuation of G. Similarly, this implies that $\lambda(x)\lambda(z)^{-1} = \lambda(u)\lambda(v)^{-1}$, contrary to λ being a Λ-valuation of H. Thus, for every non-trivial edge label induced by μ, there are exactly two edges that share that label. ■

Note that μ^* defines an equivalence relation, where our equivalence classes are the set of edges sharing a common label.

Definition 5.2.3 *Let G and H be graphs and let Γ and Λ be groups. Suppose that G has a Γ-valuation, γ, and that H has a Λ-valuation, λ. Let μ be a product valuation as in Proposition 5.2.2. We define a* subproduct *of G and H as a graph with vertex set $V(G) \times V(H)$ and*

Figure 5.3 Various Subproducts of P_2 and P_3



whose edge set is derived from that of $G \boxtimes H$ by retaining exactly one edge from each of the equivalence classes induced by μ^*.

Theorem 5.2.4 *Let G and H be graphs and let Γ and Λ be groups. Suppose that G has a closed Γ-valuation, γ, and that H has a closed Λ-valuation, λ. The mapping μ, as defined in Proposition 5.2.2, is a closed $(\Gamma \times \Lambda)$-valuation of any subproduct of G and H induced by μ. Thus, any subproduct of G and H induced by μ is an automorphic divisor.*

Proof.

Let $P_\mu(G, H)$ be an arbitrary subproduct of G and H induced by μ. Note that μ is a $(\Gamma \times \Lambda)$-labelling of $P_\mu(G, H)$ by Proposition 5.2.1. Since $P_\mu(G, H)$ is obtained from $G \boxtimes H$ by retaining exactly one edge from each equivalence class defined by μ^*, it follows that μ^* is an injective mapping by construction. As such, μ is a $(\Gamma \times \Lambda)$-valuation of $P_\mu(G, H)$. It remains to be shown that μ is a closed $(\Gamma \times \Lambda)$-valuation of $P_\mu(G, H)$.

Let $(s, u), (t, v) \in V(P_\mu(G, H))$ and note that:

$$\mu^*((s, u), (t, v)) = (\gamma(s)\gamma(t)^{-1}, \lambda(u)\lambda(v)^{-1}).$$

Since γ is a Γ-valuation of G, it follows that $\gamma(s)\gamma(t)^{-1}$ is an edge label of G. Thus, $\gamma(s)\gamma(t)^{-1}$ is not an involution in Γ by definition of γ. A similar argument shows that $\lambda(u)\lambda(v)^{-1}$ is not an involution in Λ. From this it follows that none of the edge labels are involutions. Further, by definition of $G \boxtimes H$, for all $u, v \in H$ we have that $(s, u)(t, v) \in E(G \boxtimes H)$. Thus, the label $\mu^*((s,u),(t,v))$ appears exactly once as an edge label of $P_\mu(G,H)$. Hence, μ is a closed $(\Gamma \times \Lambda)$-valuation of $P_\mu(G,H)$. Ergo, $P_\mu(G,H)$ is an automorphic divisor by Theorem 5.1.6. ■

Note that the subproduct is not necessarily unique. Figure 5.3 gives several non-isomorphic examples of the subproduct of P_2 and P_3 generated by their graceful labellings. Further note, if there are k pairs of non-trivial edge labels, we have 2^k possible choices for how to construct the new graph. Further, for each edge with the label $(a,1)$, we have $n(G)$ choices of which $n(G)-1$ edges to delete. Similarly, for each edge with label $(1,b)$, we have $n(H)$ choices of which $n(H)-1$ edges to delete. However, some of the graphs generated by this deletion of edges may be isomorphic.

Similarly, we examine the question of whether the join of two automorphic divisors is also an automorphic divisor. We hope to use a method similar to what was used to construct subproducts. However, this requires a bit more machinery.

Lemma 5.2.5 *Let G be a graph and Γ be a group.*

(i) If G has a Γ-labelling, then $n(G) \leq o(\Gamma)$.

(ii) If G has a Γ-valuation, then it follows that $o(\Gamma) \geq 2e(G) + i(\Gamma)$.

(iii) If G is a connected graph with a Γ-valuation, then it follows that $n(G) \leq \frac{o(\Gamma)-i(\Gamma)-2}{2}$.

(iv) If G is a graph with no isolated vertices and has a closed Γ-valuation γ, then it follows that $n(G) < o(\Gamma)$.

Proof.

(i) Since G has a Γ-labelling, it follows that any two distinct vertices of G have different labels. Because all of the vertex labels must come from Γ, we must have that $o(\Gamma) \geq n(G)$.

(ii) Since any Γ-valuation of G is a Γ-labelling, we must have $n(G) \leq o(\Gamma)$ by (i). Further, each edge must receive a different label from Γ. The edge labels and their inverses must all be distinct elements of Γ. Moreover, none of these elements may be involutions in Γ. Thus, it follows that $o(\Gamma) \geq 2e(G) + i(\Gamma)$.

(iii) Since G is connected, it follows that $n(G) \leq e(G) - 1$ by Proposition 1.1.6. Further, because G has a Γ-valuation it follows from (ii) that:

$$\begin{aligned} o(\Gamma) &\geq 2e(G) + i(\Gamma) \\ \Rightarrow \frac{o(\Gamma) - i(\Gamma)}{2} &\geq e(G) \geq n(G) + 1 \\ \Rightarrow n(G) &\leq \frac{o(\Gamma) - i(\Gamma) - 2}{2}. \end{aligned}$$

(iv) Since γ is a Γ-labelling, we must have that $n(G) \leq o(\Gamma)$ by (i). Thus, we need only show that $n(G) \neq o(\Gamma)$. Suppose to the contrary that $n(G) = o(\Gamma)$. Because G has a closed Γ-valuation, it follows from Theorem 5.1.6 that there exists a set of differences S such that G is an automorphic divisor of $H = C_\Gamma(S)$. Note that $n(H) = o(\Gamma) = n(G)$. Thus, every G-block in the automorphic decomposition $\mathscr{D}$ must intersect every other block. This implies that $I(\mathscr{D}) \cong K_{n(G)}$. Since $\mathscr{D}$ is automorphic, it follows that $H \cong K_{n(G)}$. Theorem 3.2.1 implies that

$$e(H) = \frac{n(G)(n(G)-1)}{2} \quad \text{and} \quad e(G) = \frac{n(G)-1}{2}.$$

Thus, the number of edges in G is less than half the number of vertices of G. It follows from Proposition 1.1.6 that G has at least one isolated vertex. Hence, we have a contradiction. Thus $n(G) < o(\Gamma)$.

■

In particular, Lemma 5.2.5 implies that if G is a graph with no isolated vertices and Γ is a group such that G has a closed Γ-valuation, γ, then it follows that $\exists u \in \Gamma$ such that for all $x \in V(G)$ we have that $\gamma(x) \neq u$. Further, since $2n(G) < o(\Gamma) - i(\Gamma)$ by Lemma

5.2.5, we can assume that u can be chosen such that $u \neq u^{-1}$. By taking advantage of these unused elements, we hope to construct the required valuation.

Lemma 5.2.6 *Let G and H be connected graphs and let Γ and Λ be groups. Suppose that G has a Γ-valuation, γ and that H has a Λ-valuation, λ. Then there exists a $(\Gamma \times \Lambda)$-labelling of $G \vee H$.*

Proof.

Since G and H have valuations, it follows from Lemma 5.2.5 that there are $u \in \Gamma$ and $w \in \Lambda$ such that for all $g \in V(G)$ and $h \in V(H)$, $\gamma(g) \neq u$ and $\lambda(h) \neq w$. Further, as $2n(G) \leq o(\Gamma) - i(\Gamma) - 2$, we may suppose that u is not an involution in Γ. Similarly, we may assume that w is not an involution in Λ. We define $\mu : V(G) \cup V(H) \to \Gamma \times \Lambda$ by:

$$\mu(x) = \begin{cases} (\gamma(x), w) & \text{if} \quad x \in V(G) \\ (u, \lambda(x)) & \text{if} \quad x \in V(H). \end{cases}$$

We claim that μ is the required mapping.

First, we show that μ is injective. If $x, y \in V(G)$ and $\mu(x) = \mu(y)$, then $(\gamma(x), w) = (\gamma(y), w)$. This implies that $\gamma(x) = \gamma(y)$. Since γ is injective, it follows that $x = y$. The case where $x, y \in V(H)$ follows by the same argument. If $x \in V(G)$, $y \in V(H)$, and $\mu(x) = \mu(y)$, then we have $(\gamma(x), w) = (u, \lambda(y))$. This implies that $\gamma(x) = u$ (contrary to definition of u) and that $\lambda(y) = w$ (contrary to definition of w). Thus, μ is a $(\Gamma \times \Lambda)$-labelling of $G \vee H$. ■

In order to show that μ is a valuation, we must show that μ^* is injective and that the induced edge mapping contains no involutions. We begin by showing that $\mu^*(E(G \vee H))$ contains no involutions.

Lemma 5.2.7 *Let G and H be graphs and let Γ and Λ be groups. Suppose that G has a Γ-valuation and that H has a Λ-valuation. If μ is defined as in Lemma 5.2.6, then $\mu^*(E(G \vee H))$ contains no involutions in $\Gamma \times \Lambda$.*

Proof.

Take $xy \in E(G \vee H)$. If $x, y \in V(G)$ then $\mu^*(xy) = (\gamma(x)\gamma(y)^{-1}, 1)$. Note that this is an involution in $\Gamma \times \Lambda$ if and only if $\gamma(x)\gamma(y)^{-1}$ is an involution in Γ. However, $\gamma(x)\gamma(y)^{-1}$ cannot be an involution in Γ as γ is a Γ-valuation on G. A similar argument holds for the case where $x, y \in V(H)$.

Thus, we may suppose that $x \in V(G)$ and $y \in V(H)$. Note that:

$$\mu^*(xy) = (\gamma(x)u^{-1}, w\lambda(y)^{-1}).$$

This is an involution if and only if $\gamma(x)u^{-1}$ and $w\lambda(y)^{-1}$ are both involutions in their respective groups.

It suffices to show that $\gamma(x)u^{-1}$ is not an involution in Γ. Suppose to the contrary that $\gamma(x)u^{-1}$ is an involution in Γ. This implies that $\gamma(x)u^{-1} = u\gamma(x)^{-1}$, or equivalently, $u = \gamma(x)u^{-1}\gamma(x)$. If there is no $v \in V(G)$ such that $\gamma(v) = 1$, then we may define a new mapping γ' by $\gamma'(x) = \gamma(x)\gamma(v)^{-1}$. Thus, we may assume that there is a $v \in V(G)$ such that $\gamma(v) = 1$. Since x was chosen arbitrarily, this holds for all $x \in V(G)$. In particular, this holds for $v \in V(G)$. As such, $u = u^{-1}$, a contradiction. Thus, no edges in $G \vee H$ are labelled with involutions. ■

Now we show that μ^* is also injective. This requires several cases:

Case 1: Assume that $xy, st \in E(G)$ and $\mu^*(xy) = \mu^*(st)$. This implies that $\gamma^*(xy) = \gamma^*(st)$. Since γ^* is injective, it follows that $xy = st$. The case where $xy, st \in E(H)$ follows by the same argument.

Case 2: Assume that $xy \in E(G)$ and $st \in E(H)$. If $\mu^*(xy) = \mu^*(st)$, then:

$$(\gamma(x)\gamma(y)^{-1}, 1) = (1, \lambda(s)\lambda(t)^{-1}).$$

Thus, $\gamma(x)\gamma(y)^{-1} = 1$ or equivalently $\gamma(x) = \gamma(y)$. Thus by injectivity of γ, we have that $x = y$. Similarly, $\lambda(s)\lambda(t)^{-1} = 1$ implies that $s = t$. Since we do not allow our graphs to have loops, this case is impossible.

Case 3: Assume that $xy \in E(G), s \in V(G)$, and $t \in V(H)$. By definition, we have that $xy, st \in E(G \vee H)$. Let:

$$\mu^*(xy) = \mu^*(st)$$

$$\Rightarrow (\gamma^*(xy), 1) = (\gamma(s)u^{-1}, w\lambda(t)^{-1})$$

$$\Rightarrow w\lambda(t)^{-1} = 1 \Rightarrow \lambda(t) = w.$$

This contradicts the definition of w. The case where $xy \in E(H), s \in V(G)$, and $t \in V(H)$ follows analogously.

Case 4: Assume that $x, s \in V(G)$ and $y, t \in V(H)$. By definition, we have that $xy, st \in E(G \vee H)$. Let:

$$\mu^*(xy) = \mu^*(st).$$

This implies that either

$$(\gamma(x)u^{-1}, w\lambda(y)^{-1}) = (\gamma(s)u^{-1}, w\lambda(t)^{-1}) \quad \text{or}$$

$$(u\gamma(x)^{-1}, \lambda(y)w^{-1}) = (\gamma(s)u^{-1}, w\lambda(t)^{-1}).$$

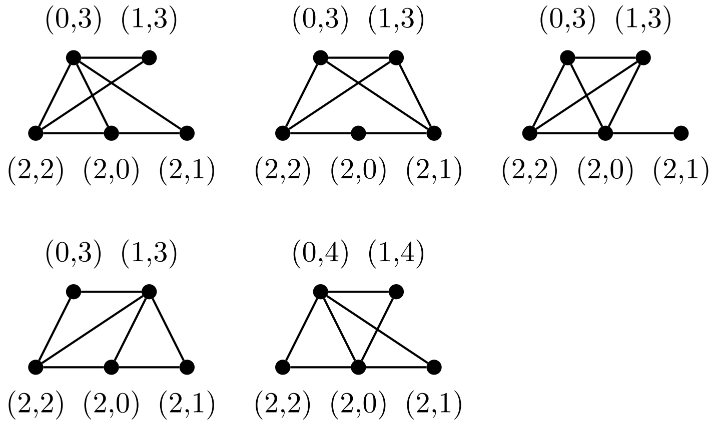
If $(\gamma(x)u^{-1}, w\lambda(y)) = (\gamma(s)u^{-1}, w\lambda(t)^{-1})$, then $\gamma(x)u^{-1} = \gamma(s)u^{-1}$. This implies that $\gamma(x) = \gamma(s)$. Thus by injectivity of γ, we have that $x = s$. Similarly, $w\lambda(y) = w\lambda(t)$ implies that $y = t$. Thus we have that $xy = st$.

If $(u\gamma(x)^{-1}, \lambda(y)w^{-1}) = (\gamma(s)u^{-1}, w\lambda(t)^{-1})$, then $u = \gamma(s)u^{-1}\gamma(x)$ and $\lambda(y) = w\lambda(t)w$. Thus, we may not have injectivity in this case.

Note that in the last case it was possible to obtain duplicate edge labels. To remedy this, we use the same tactic we used in constructing the subproduct in the previous section. Namely, we delete edges from the join such that μ is a $(\Gamma \times \Lambda)$-valuation of the resulting graph.

Definition 5.2.8 *Let G and H be graphs and let Γ and Λ be groups. Suppose that G has a Γ-valuation, γ, and that H has a Λ-valuation, λ. Define μ as in Lemma 5.2.6. A* subjoin *of G and H induced by μ is a graph with vertex set $V(G) \cup V(H)$ and whose edge set is derived*

Figure 5.4 Subjoins of P_2 and P_3



from that of $G \vee H$ by retaining exactly one edge from each of the equivalence classes defined by μ^*.

Note that the subjoin of two graphs is not unique. Figure 5.4 gives several examples of subjoins of P_2 and P_3 generated by their graceful labelling. We may also obtain different subjoins by changing the unused element. This is illustrated by the last graph in Figure 5.4.

Theorem 5.2.9 *Let G and H be graphs with no isolated vertices and let Γ and Λ be groups. If G has a closed Γ-valuation, γ, and H has a closed Λ-valuation, λ, then any subjoin of G and H has a closed $(\Gamma \times \Lambda)$-valuation.*

Proof.

Since G and H have no isolated vertices it follows from Lemma 5.2.5 that there are $u \in \Gamma$ and $w \in \Lambda$ such that for all $g \in V(G)$ and $h \in V(H)$, $\gamma(g) \neq u$ and $\lambda(h) \neq w$. Define $\mu : V(G) \cup V(H) \to \Gamma \times \Lambda$ by:

$$\mu(x) = \begin{cases} (\gamma(x), w) & \text{if} \quad x \in V(G) \\ (u, \lambda(x)) & \text{if} \quad x \in V(H). \end{cases}$$

Let $J_\mu(G, H)$ be an arbitrary subjoin induced by μ. We claim that μ is the required mapping. By Lemma 5.2.6, μ is injective on the vertex set of $G \vee H$. Further, by Lemma 5.2.7, $\mu^*(E(G \vee H))$ contains no involutions in $\Gamma \times \Lambda$. Note that μ^* is injective on the edge set of $J_\mu(G, H)$ by construction of the subjoin. Thus, μ is a $(\Gamma \times \Lambda)$-valuation on $J_\mu(G, H)$.

We must also show that this mapping is closed. Let $x, y \in V(G \vee H)$. If $x \in V(G)$ and $y \in V(H)$, then $xy \in E(G \vee H)$ by definition. Thus, $\mu^*(xy)$ is an edge label in $J_\mu(G, H)$. So we may assume without loss of generality that $x, y \in V(G)$. Note that $\mu^*(x, y) = (\gamma^*(x, y), 1)$. Since γ is a closed Γ-valuation, it follows that there exists $st \in E(G)$ such that $\gamma^*(x, y) = \gamma^*(st)$. By definition, $st \in E(G \vee H)$. Thus, $\mu^*(x, y) = \mu^*(st)$. As such, μ is a closed $(\Gamma \times \Lambda)$-valuation of $J_\mu(G, H)$. ■

Remark 5.2.10 *We have always assumed that our prototypes had no isolated nodes in order to avoid a degenerate case. However, it may be valuable to determine which of these graphs have closed valuations. This would allow us to join them to another graph. Since the resulting joined graph would have no isolated nodes, it would be perfectly acceptable as a prototype. In such cases, we will always assume that the group is large enough to accommodate the required valuation. This assures that the proof of Theorem 5.2.9 is still valid.*

Theorem 5.2.11 *Let G, H be graphs and let Γ, Λ be groups such that $o(\Gamma) - i(\Gamma) > n(G)$ and $o(\Lambda) - i(\Lambda) > n(H)$. If G has a closed Γ-valuation, γ, and H has a closed Λ-valuation, λ, then the subjoin of G and H has a closed $(\Gamma \times \Lambda)$-valuation.*

Proof.

Note that $n(G) < o(\Gamma) - i(\Gamma)$ and $n(H) < o(\Lambda) - i(\Lambda)$. Further, the appropriate valuations exist. Thus, there are non-involutory elements $u \in \Gamma$ and $w \in \Lambda$ such that for all $g \in V(G)$ and $h \in V(H)$, $\gamma(g) \neq u$ and $\lambda(h) \neq w$. The rest of the proof is analogous to the proof of Theorem 5.2.9. ■

In our analysis, we showed that when $x, s \in V(G)$ and $y, t \in V(H)$, the resulting joined graph could have repeated edge labels. However, this case requires both graphs to have at least two distinct vertices. Hence if $G \cong K_1$, then the subjoin of G and H is isomorphic to $G \vee H$. Thus, we need to establish that K_1 has a closed valuation.

Lemma 5.2.12 *Let K_1 be the graph consisting of only a single vertex. K_1 has a closed $\mathbb{Z}_3$-valuation.*

Proof.

Let x be the unique vertex of K_1. Let $\gamma(x) = 0$. We claim that this is the required valuation. Since x is the only vertex of K_1, it follows that this is a $\mathbb{Z}_3$-labelling. Since $E(K_1) = \emptyset$, it follows vacuously that γ is a $\mathbb{Z}_3$-valuation. Similarly, it follows vacuously that γ is a closed $\mathbb{Z}_3$-valuation. ■

Note that we could use $\mathbb{Z}_2$ to obtain the same result as in Lemma 5.2.12. However, as every element of $\mathbb{Z}_2$ is an involution, this would be an undesirable case for us.

Proposition 5.2.13 *Suppose that G has a closed Γ-valuation. Then $G \vee K_1$ has a closed $\Gamma \times \mathbb{Z}_3$-valuation.*

Proof.

Note that $G \vee K_1$ is isomorphic to the subjoin of G and K_1 for any μ and apply Theorem 5.2.11. ■

Proposition 5.2.13 allows us to show that graphs with a universal vertex will often have a closed valuation. In particular, this allows us to give a different proof that complete graphs have closed valuations.

Lemma 5.2.14 *For $n \geq 2$, $K_n \cong K_{n-1} \vee K_1$.*

Proof.

It suffices to show that the vertices of $K_{n-1} \vee K_1$ form a clique of order n. By definition, the vertices of K_{n-1} form a clique of order $n-1$. By definition of the join, for all $x \in K_{n-1}$ and $y \in K_1$, $xy \in E(K_{n-1} \vee K_1)$. Thus we have the required clique. ■

Proposition 5.2.15 *K_n has a closed $\mathbb{Z}_3^n$-valuation.*

Proof.

We proceed by induction on n. The case where $n = 1$ follows from Proposition 5.2.12. Assume that for some $n \geq 1$, K_n has a closed $\mathbb{Z}_3^n$-valuation. By Lemma 5.2.14, we have that $K_{n+1} \cong K_n \vee K_1$. By the inductive hypothesis, K_n has a closed $\mathbb{Z}_3^n$-valuation. By Proposition 5.2.12, K_1 has a closed $\mathbb{Z}_3$-valuation. Since both K_n and K_1 have closed valuations, it follows from Theorem 5.2.11, that $K_{n+1} \cong K_n \vee K_1$ has a closed $\mathbb{Z}_3^n \times \mathbb{Z}_3$-valuation. Since $\mathbb{Z}_3^{n+1} \cong \mathbb{Z}_3^n \times \mathbb{Z}_3$, the claim follows from the principle of mathematical induction. ■

It is interesting to note that even though the method of this proof differs greatly from our original proof that the complete graph is an automorphic divisor, the resulting labelling is very similar. However, we are unable to use $\mathbb{Z}_2$ in this construction because of the complications that arise due to involutions.

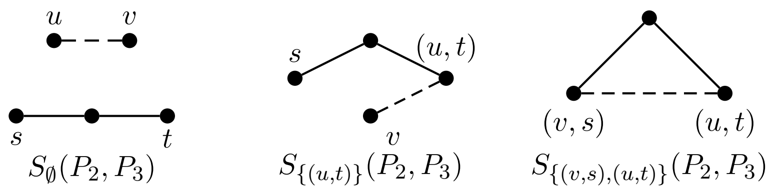
Except for the case where $\Delta(G) = n(G) - 1$, it may be difficult to determine whether a given graph can be represented as a subjoin. As with subproducts, this method is a purely constructive one.

5.3 Other Constructions

We now detail several additional methods for constructing closed valuations. Recall that we know of a closed valuation for K_p. Consider the problem of finding a closed valuation for the graph G, where G is obtained from K_p by deleting a single edge. Granted, this problem could be solved by using P_3 and K_1 and inductively applying Proposition 5.2.13 until G is obtained. However, we are looking at a slightly different principle.

For convenience of exposition, assume that f is the closed $\mathbb{Z}_{2^p-1}$-valuation on K_p defined in Theorem 2.3.11, i.e., $f(v_i) = 2^i - 1$ for $i = 0, ..., p-1$. Note that in K_p, each of the edges is interchangeable. Thus, it does not matter which edge we delete in order to obtain G. So without loss of generality, we delete the edge labelled $2^{p-1} - 1$. In other words, we delete the edge with the largest label. Now we “shrink” the size of the group so that the

Figure 5.5 Superimpositions of P_2 and P_3



valuation remains closed. Note that:

$$2^{p-1} - 2 \equiv -(2^{p-1} - 1) \pmod{2^p - 3}.$$

It is easy to confirm that the other requirements of a closed valuation are met.

While the idea of shrinking a group does not generalize easily, there are other ideas of value here. In particular, in constructing a closed valuation on a graph, it is valuable to choose a group that does not leave many unused edge labels. Hence, it is often more desirable to obtain a complete valuation than to obtain a more general closed valuation. With this observation, we proceed.

Definition 5.3.1 *Let G and H be graphs and let $A \subseteq V(G) \times V(H)$. Define*

$$A(G) = \{a \in V(G) : (a, h) \in A \ \textit{ for some } \ h \in V(H)\} \quad \textit{and}$$

$$A(H) = \{a \in V(H) : (g, a) \in A \ \textit{ for some } \ g \in V(G)\}.$$

The superimposition *of G and H induced by A, denoted $S_A(G, H)$, is the graph with:*

$$V(S_A(G, H)) = V(G) \cup V(H) - A(G) \quad \textit{and}$$

$$xy \in E(S_A(G, H)) \quad \textit{if and only if} \quad xy \in E(G) \quad \textit{or} \quad xy \in E(H).$$

Examples of superimpositions of P_2 and P_3 are given in Figure 5.5.

Given graphs G and H that accept closed valuations, we would be interested in the problem of whether $S_A(G, H)$ admits a closed valuation for various choices of A. Note that in the special case where G is a tree and $H \cong P_2$, showing that $S_{\{(u,v)\}}(G, P_2)$ admits a closed valuation would be very close to proving Ringel's Conjecture. As such, we satisfy ourselves with specific examples. To do this, we define several labellings on graphs.

Definition 5.3.2 *Let G be a graph of order p and size q and let $k, \ell \in \mathbb{N}$ be such that $k, \ell \geq q$. Suppose that f is an injective mapping from $V(G)$ to $\{0, 1, ..., \ell\}$. Define the induced edge mapping by $f^*(x, y) = |f(x) - f(y)|$. If f^* is an injective mapping of $E(G)$ to $\{1, 2, ..., q-1, k\}$, then f is a (ℓ, k)*-sequential labelling.

We note that by definition, a graceful labelling of G is a (q, q)-sequential valuation of G. Further, a (ℓ_1, k)-sequential valuation of G is a (ℓ_2, k)-sequential valuation of G for all $\ell_2 \geq \ell_1$. As such, we will assume that if f is a (ℓ, k)-sequential valuation, then $\ell = \max_{v \in V(G)} f(v)$.

Theorem 5.3.3 *Let G be a graph of size q and take $\ell \leq 2q$.*

(i) If f is a (ℓ, q)-sequential valuation of G, then f is a complete $\mathbb{Z}_{2q+1}$-valuation of G.

(ii) If f is a $(\ell, q+1)$-sequential valuation of G, then f is a complete $\mathbb{Z}_{2q+1}$-valuation of G.

Proof.

(i) Since f is an injective mapping from $V(G) \to \{0, 1, ..., \ell\} \subseteq \{0, 1, ..., 2q\}$, it follows that f is a $\mathbb{Z}_{2q+1}$-labelling of G. Further, since f^* is an injective mapping from $E(G)$ to $\{1, 2, ..., q-1, q\}$, it follows that all of the edge labels and their inverses are distinct modulo $2q+1$. Thus, f is a $\mathbb{Z}_{2q+1}$-valuation of G. Since the edge labels (and their inverses) are all the non-identity elements of the group $\mathbb{Z}_{2q+1}$, it follows that for any $x, y \in V(G)$ where $x \neq y$, we have that $f^*(x, y) \in f^*(E(G))$. Thus f is a closed valuation, moreover, it is complete.

(ii) Since f is an injective mapping from $V(G) \to \{0, 1, ..., \ell\} \subseteq \{0, 1, ..., 2q\}$, it follows that f is a $\mathbb{Z}_{2q+1}$-labelling of G. Further, since f^* is an injective mapping from $E(G)$ to $\{1, 2, ..., q-1, q+1\}$, it follows that all of the edge labels and their inverses are distinct modulo $2q+1$. Note that $q+1 \equiv -q \pmod{2q+1}$, thus f is a $\mathbb{Z}_{2q+1}$-valuation of G. Since the edge labels (and their inverses) are all the non-identity elements of the group $\mathbb{Z}_{2q+1}$, it follows that for any $x, y \in V(G)$ where $x \neq y$, we have that $f^*(x, y) \in f^*(E(G))$. Thus f is a closed valuation, moreover, it is complete.

■

Theorem 5.3.3 implies that we are primarily interested in (ℓ, k)-sequential valuations for certain values of ℓ and k. These valuations are summarized in the next definition.

Definition 5.3.4 *Let G be a graph of size q. For $\ell \leq 2q$, a (ℓ, q)-sequential valuation of G will be called a q-*sequential valuation *of G. For $\ell \leq 2q$, a $(\ell, q+1)$-sequential valuation of G will be called a* partial q-sequential valuation *of G.*

We will use this definition, as well as our notion of a superimposition, to construct new examples. In particular, we are able to use these concepts to generalize Rosa's result that any graph with a graceful base has a ρ-valuation [66].

Theorem 5.3.5 *Let $G = (V, E)$ be a graph of size q such that $u_1, ..., u_k \in V$. Let $a_1, ..., a_k \in V(K_{1,k})$ be such that $deg_{K_{1,k}}(a_i) = 1$ for $i = 1, ..., k$. Take $a \in V(K_{1,k})$ such that $deg_{K_{1,k}}(a) = k$. Define $A = \{(u_i, a_i) : i = 1, ..., k\}$ and $G' = S_A(G, K_{1,k})$.*

(i) Let f be a q-sequential valuation of G such that $f(u_i) = m - i + 1$ for $i - 1, ..., k$ and $m + q + 1 \pmod{2(q+k)+1} \notin f(V)$. Then G' has a $(q+k)$-sequential valuation.

(ii) Let f be a partial q-sequential valuation of G such that $f(u_1) = m$, $f(u_i) = m - i + 1$ for $i \geq 2$, and $m + q \pmod{2(q+k)+1} \notin f(V)$. Then G' has a $(q+k)$-sequential valuation.

(iii) Let f be a q-sequential valuation of G such that $f(u_i) = m - i + 1$ for $i = 1, ..., k-1$, $f(u_k) = m - k$, and $m + q + 1 \pmod{2(q+k)+1} \notin f(V)$. Then G' has a partial $(q+k)$-sequential valuation.

(iv) Let f be a partial q-sequential valuation of G such that $f(u_1) = m$, $f(u_i) = m - i$ for $i = 2, ..., k-1$, $f(u_k) = m - k - 1$, and $m + q \pmod{2(q+k)+1} \notin f(V)$. Then G' has a partial $(q+k)$-sequential valuation.

Proof.

Note that in all proofs $e(G') = q + k$.

(i) It suffices to give the appropriate $(q+k)$-sequential valuation. Define:

$$g : V(G') \to \{0, ..., 2(q+k)\}$$

as follows:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V \\ m + q + 1 & \text{if} \quad x = a. \end{cases}$$

Since f is an injective mapping on V and $m + q + 1 \pmod{2(q+k)+1} \notin f(V)$, it follows that g is an injective mapping on $V(G')$. Since f is q-sequential, it follows that $\{1, ..., q\} \subseteq g^*(E(G'))$. By definition of G', the new edges are of the form au_i for $i = 1, ..., k$. The edge labels induced by g are:

$$g^*(au_i) = |g(a) - g(u_i)| = m + q + 1 - (m - i + 1) = q + i.$$

Note that these edge labels are distinct for all i. Further, since the largest edge label induced by f is q, it follows that g^* is injective. Further, the induced edge labels are $g^*(E(G')) = \{1, ..., q+k\}$. Thus, g is a $(q+k)$-sequential valuation by definition.

(ii) It suffices to give the appropriate $(q+k)$-sequential valuation. Define

$$g : V(G') \to \{0, ..., 2(q+k)\}$$

as follows:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V \\ m+q & \text{if} \quad x = a. \end{cases}$$

Since f is an injective map on V and $m+q \pmod{2(q+k)+1} \notin f(V)$, it follows that g is an injective mapping on $V(G')$. Since f is a partial q-sequential valuation, it follows that $\{1, ..., q-1, q+1\} \subseteq g^*(E(G'))$. By definition of G', the new edges are of the form au_i for $i = 1, ..., k$. Note that:

$$g^*(au_1) = |g(a) - g(u_i)| = m + q - m = q.$$

Further for $i = 2, ..., k$, we have that:

$$g^*(au_i) = |g(a) - g(u_i)| = m + q + 1 - (m - i + 1) = q + i.$$

Note that these edge labels are distinct for all i. Since the edge labels induced by f are $\{1, ..., q-1, q+1\}$, it follows that g^* is injective. Since the induced edge labels are $g^*(E(G')) = \{1, ..., q+k\}$, it follows that g is a $(q+k)$-sequential valuation by definition.

(iii) It suffices to give the appropriate partial $(q+k)$-sequential valuation. Define

$$g : V(G') \to \{0, ..., 2(q+k)\}$$

as follows:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V \\ m+q+1 & \text{if} \quad x = a. \end{cases}$$

Since f is an injective map on V and $m+q+1 \pmod{2(q+k)+1} \notin f(V)$, it follows that g is an injective mapping on $V(G')$. Since f is q-sequential, it follows that $\{1, ..., q\} \subseteq g^*(E(G'))$. By definition of G', the new edges are of the form au_i for $i = 1, ..., k$. The edge labels induced by g for $i = 1, ..., k-1$ are:

$$g^*(au_i) = |g(a) - g(u_i)| = m + q + 1 - (m - i + 1) = q + i.$$

Also,

$$g^*(au_k) = |g(a) - g(u_k)| = m + q + 1 - (m - k) = q + k + 1.$$

Note that these edge labels are distinct for all i. Further, since the largest edge label induced by f is q, it follows that g^* is injective. Further, the induced edge labels are $g^*(E(G')) = \{1, ..., q+k-1, q+k\}$. Thus, g is a partial $(q+k)$-sequential valuation by definition.

(iv) It suffices to give the appropriate partial $(q+k)$-sequential valuation. Define

$$g : V(G') \to \{0, ..., 2(q+k)\}$$

as follows:

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V \\ m+q & \text{if} \quad x = a. \end{cases}$$

Since f is an injective map on V and $m + q \pmod{2(q+k)+1} \notin f(V)$, it follows that g is an injective mapping on $V(G')$. Since f is a partial q-sequential valuation, it follows that $\{1, ..., q-1, q+1\} \subseteq g^*(E(G'))$. By definition of G', the new edges are of the form au_i for $i = 1, ..., k$. The edge labels induced by g are:

$$g^*(au_1) = |g(a) - g(u_1)| = m + q - m = q \quad \text{and}$$

$$g^*(au_k) = |g(a) - g(u_k)| = m + q - (m - k - 1) = q + k + 1.$$

For $i = 2, ..., k-1$ we have that:

$$g^*(au_i) = |g(a) - g(u_i)| = m + q - (m - i) = q + i.$$

Note that these edge labels are distinct for all i. Further, since the edge labels induced by f are $\{1, ..., q-1, q+1\}$, it follows that g^* is injective. Further, the induced edge labels are $g^*(E(G')) = \{1, ..., q+k-1, q+k+1\}$. Thus, g is a partial $(q+k)$-sequential valuation by definition.

■

It should be noted that this construction will lend itself to other possibilities than those listed above. However, listing all of these possibilities would be an extensive process. Moreover, these are very similar to the above construction and are thus omitted.

Corollary 5.3.6 *Let G be a graceful graph of size q with $u_1, ..., u_k \in V(G)$. Take $a_1, ..., a_k \in V(K_{1,k})$ such that $deg_{K_{1,k}}(a_i) = 1$ for $i = 1, ..., k$. If $A = \{(u_i, a_i) : i = 1, ..., k\}$, then $S_A(G, K_{1,k})$ has a $(q+k)$-sequential valuation.*

Proof.

Since G is a graceful graph of size q, it has a q-sequential valuation by definition. Thus we may apply Theorem 5.3.5 to achieve the desired result. ■

Note that given a group Γ, we can construct a graph G such that G has a closed Γ-valuation.

Theorem 5.3.7 *Suppose that Γ is a group and γ is an injective mapping that takes the elements of some n-set into the non-involutory elements of Γ. Then there exists a graph G such that γ is a closed Γ-valuation on G.*

Proof.

We proceed by constructing the required graph. Let $V(G)$ be an n-set such that γ is injective on $V(G)$. Consider all possible two element subsets of $V(G)$. Note that γ^* defines an equivalence relation, where the equivalence classes consist of all pairings that have the same induced label. We allow into the edge set of G exactly one pair from each equivalence class. Thus, γ^* is injective. Since the label of every non-edge is on some edge and we do not allow involutions, this valuation is closed. ■

Remark 5.3.8 *It should be noted that the construction described in Theorem 5.3.7 may result in a graph with one or more isolated vertices. This is not a problem because if γ is injective on an n-set, it is injective on a k-set for $k \leq n$. Also, depending on the labelling, this method may result in a complete graph.*

5.4 Persistent Automorphic Divisors

Previously in this chapter, we have looked at the problem of whether a graph G is an automorphic divisor for some graph. In particular, we studied the problem of whether G would have a closed Γ-valuation f which would imply the existence of a Γ-translational automorphic G-decomposition. In this section, we look at the problem of whether a graph G is an automorphic divisor for an infinite number of non-isomorphic connected graphs. We restrict our attention to connected graphs because if H_1 and H_2 admit an automorphic G-decomposition, then $H_1 \cup H_2$ does as well.

Proposition 5.4.1 *Suppose that H_1 and H_2 both admit an automorphic G-decomposition. Then $H_1 \cup H_2$ admits an automorphic G-decomposition.*

Proof.

Let $\mathscr{D}_1$ be an automorphic G-decomposition of H_1. Let $\mathscr{D}_2$ be an automorphic G-decomposition of H_2. By definition, $\mathscr{D}_1$ is a G-decomposition of H_1 and $\mathscr{D}_2$ is a G-decomposition of H_2. Thus, $\mathscr{D}_1 \cup \mathscr{D}_2$ is an edge decomposition of $H_1 \cup H_2$, by definition of union of graphs. Note that each part of this partition is isomorphic to G. Thus, $\mathscr{D}_1 \cup \mathscr{D}_2$ is a G-decomposition of $H_1 \cup H_2$. Further note, $I(\mathscr{D}_1) \cong H_1$ and $I(\mathscr{D}_2) \cong H_2$, by definition of automorphic decomposition. Since $\mathscr{D}_1$ and $\mathscr{D}_2$ decompose disjoint graphs, they are node disjoint. Thus, $I(\mathscr{D}_1 \cup \mathscr{D}_2) \cong I(\mathscr{D}_1) \cup I(\mathscr{D}_2) \cong H_1 \cup H_2$. Thus, $\mathscr{D}_1 \cup \mathscr{D}_2$ is an automorphic G-decomposition of $H_1 \cup H_2$. ■

A simple way of insuring that the graphs we obtain are non-isomorphic is to have them be of different order. Thus, if G admits a closed $\mathbb{Z}_n$-valuation f we wish to know if G admits a closed $\mathbb{Z}_m$-valuation g where $m \geq n$. In particular, we wish to know if f is a closed $\mathbb{Z}_m$-valuation for all $m \geq n$. If f is a $\mathbb{Z}_m$-valuation for all $m \geq n$, then we say that f is a *persistent* $\mathbb{Z}_n$*-valuation*. If f is a closed $\mathbb{Z}_m$-valuation for all $m \geq n$, then we say that f is a *closed persistent* $\mathbb{Z}_n$*-valuation*. By convention, if we say f is a (closed) persistent $\mathbb{Z}_n$-valuation, then n is the smallest integer for which f is a (closed) persistent $\mathbb{Z}_n$-valuation.

In general, valuations are not persistent as $\mathbb{Z}_m$ does not inherit the algebraic structure of $\mathbb{Z}_n$. There are basically four reasons why a closed valuation may fail to be persistent:

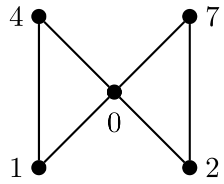
(i) The vertex labels on G are not unique in the larger modulus. For example, $0, 1, 10$ is a $\mathbb{Z}_7$-labelling of K_3, but not a $\mathbb{Z}_9$-labelling of K_3 as 1 and 10 are the same label modulo 9. While an interesting example, this case will be ignored in the future as it is unclear why one would label the vertices with elements larger than the modulus. It may be interesting to consider a labelling that changes as the modulus increases, i.e., have one or more of our vertex labels be dependent on n. This raises a philosophical question as to whether these labellings are distinct for the purposes of our discussion. Further, the existence of such a labelling is unknown. As such, we ignore this possibility.

(ii) The edge labels on G are not unique in the larger modulus. For example, $8, 1, 0, 4$ is a $\mathbb{Z}_{10}$-valuation of P_4 but not a $\mathbb{Z}_{11}$-valuation, as the edges 81 and 04 both have the label four in $\mathbb{Z}_{11}$.

(iii) An edge label on G is an involution in a larger modulus. Consider the $\mathbb{Z}_{13}$-valuation of the graph given in Figure 5.6. The edge 07 has an edge label of six modulo thirteen, but an edge label of seven modulo fourteen. Since we do not allow an edge to be labelled with an involution, this valuation is not a persistent $\mathbb{Z}_{13}$-valuation.

(iv) The difference between two non-adjacent vertices is not represented as an edge label in the larger modulus. For example, consider the graph given in Figure 5.6. Presented is a closed $\mathbb{Z}_{13}$-valuation. The vertices labelled 1 and 7 are non-adjacent and have a label of six modulo thirteen. However, there is no edge labelled six in a modulus larger than thirteen.

Figure 5.6 A Closed $\mathbb{Z}_{13}$-Valuation of the "Bowtie"



First, we consider the problem of whether a $\mathbb{Z}_n$-valuation of G will be persistent.

Lemma 5.4.2 *f is a persistent $\mathbb{Z}_n$-valuation of G if and only if for all $uv \in E(G)$ we have that $|f(u) - f(v)| < \frac{n}{2}$.*

Proof.

Suppose that f is a $\mathbb{Z}_n$-valuation such that $|f(u) - f(v)| < \frac{n}{2}$ for all $uv \in E(G)$. Let $m \geq n$. Since f is a $\mathbb{Z}_n$-labelling, it follows that f is a $\mathbb{Z}_m$-labelling. Since $\mathbb{Z}_n^+ \subseteq \mathbb{Z}_m^+$ for all $m \geq n$ and f^* is an injective map on the edge set, it follows that f is also a positive $\mathbb{Z}_m$-valuation. Hence, f is persistent.

Suppose that f is a $\mathbb{Z}_n$-valuation of G such that for some $uv \in E(G)$ we have that $|f(u) - f(v)| \geq \frac{n}{2}$. Let:

$$|f(u) - f(v)| = k \geq \frac{n}{2}.$$

If $k = \frac{n}{2}$, then $|f(u) - f(v)|_{2k} = k$. This edge label is an involution in the new modulus. Since $2k > n$ by hypothesis, this valuation is not persistent. ∎

This will aid us in determining which closed valuations are also persistent closed valuations.

Theorem 5.4.3 *Let f be a closed $\mathbb{Z}_n$-valuation of G. f is a closed persistent $\mathbb{Z}_n$-valuation of G if and only if $f(u) < \frac{n}{2}$ for all $u \in V(G)$.*

Proof.

By Lemma 5.4.2, we know that f, a $\mathbb{Z}_n$-valuation of G, is a persistent valuation if and only if $|f(u) - f(v)| < \frac{n}{2}$ for all $uv \in E(G)$.

Suppose that f is a closed persistent $\mathbb{Z}_n$-valuation of G. Without loss of generality, we may assume that there exists $u \in V(G)$ such that $f(u) = 0$. If not, we may define a new mapping $g(u) = f(u) - k$ where k is the smallest vertex label. Assume to the contrary that there is a $v \in V(G)$ such that $f(v) \geq \frac{n}{2}$. Suppose that $f(v) = \frac{n}{2}$. Since f is a closed $\mathbb{Z}_n$-valuation, then there exists a unique edge of G labelled with $|f(u) - f(v)| = \frac{n}{2}$. Since $\frac{n}{2}$ is an involution in $\mathbb{Z}_n$, this contradicts the definition of a valuation. Thus, we may assume that $f(v) = k > \frac{n}{2}$. If $uv \in E(G)$, then $|f(u) - f(v)| = k > \frac{n}{2}$, contrary to Lemma 5.4.2. By Lemma 5.4.2, no edge may be labelled k. However, since f is closed, there is a unique $xy \in E(G)$ such that $|f(x) - f(y)| = n - k$. Since f is persistent and closed, for all $m \geq n$, there must be a unique $x_m y_m \in E(G)$ such that $|f(x_m) - f(y_m)| = m - k$. However, there are infinitely many such m and only finitely many edges in G, hence, a contradiction. Thus, if f is a closed persistent $\mathbb{Z}_n$-valuation of G, then $f(u) < \frac{n}{2}$ for all $u \in V(G)$.

Suppose that f is a closed $\mathbb{Z}_n$-valuation of G such that $f(u) < \frac{n}{2}$ for all $u \in V(G)$. This implies that $|f(u) - f(v)| < \frac{n}{2}$ for all $u, v \in V(G)$. Thus:

$$|f(u) - f(v)| \subseteq \mathbb{Z}_n^+ \quad \text{for all} \quad u, v \in V(G).$$

Note that $\mathbb{Z}_n^+ \subseteq \mathbb{Z}_m^+$ for all $m \geq n$. Since f is a closed $\mathbb{Z}_n$-valuation, for all $uv \notin E(G)$ there is a unique $xy \in E(G)$ such that $|f(u) - f(v)| = |f(x) - f(y)|$. Since $|f(x) - f(y)| < \frac{n}{2} < \frac{m}{2}$ for all $m \geq n$, it follows that this edge label is constant in any modulus larger than n. Hence, f is a closed $\mathbb{Z}_m$-valuation. It follows that f is a closed persistent $\mathbb{Z}_n$-valuation. ■

We now present several examples of closed persistent valuations.

Proposition 5.4.4 *Let G be a graph and f a valuation on G. f is a closed persistent $\mathbb{Z}_n$-valuation of G if any of the following hold:*

(i) f is a graceful labelling of G and $n = 2e(G) + 1$.

(ii) G is a complete graph and $n > 2k$, where k is the largest vertex label of G.

(iii) There is a closed persistent $\mathbb{Z}_\ell$-valuation g such that for some $m \in \mathbb{Z}^+$, $f(u) = mg(u)$ for all $u \in V(G)$ and $n > m\ell$.

Proof.

(i) Any graceful labelling of G is a closed $\mathbb{Z}_{2e(G)+1}$-valuation of G. Further, by definition of graceful labelling, all of the vertex labels are less than $e(G)$. It follows by Theorem 5.4.3 that f is a closed persistent $\mathbb{Z}_{2e(G)+1}$-valuation of G.

(ii) Any valuation of K_p is trivially closed. Without loss of generality, we may assume that there is a vertex of K_p labelled 0. Thus, the largest edge label is equal to the largest vertex label. Hence, if we choose $n > 2k$ where k is the largest vertex label, then by Theorem 5.4.3, f is a closed persistent $\mathbb{Z}_n$-valuation of K_p.

(iii) Let g be a closed persistent $\mathbb{Z}_\ell$-valuation of G. Let $m \in \mathbb{Z}^+$ and define $f(u) = mg(u)$ for all $u \in V(G)$. Since the edge labels induced by g are unique, multiplying by a positive integer will not alter this. Further, if $uv \notin E(G)$, then

$$|f(u) - f(v)| = m|g(u) - g(v)| \in mg(E(G)) = f(E(G)).$$

Thus, f is closed. Since g is a closed persistent $\mathbb{Z}_\ell$-valuation, it follows from Theorem 5.4.3 that $g(u) < \frac{\ell}{2}$ for all $u \in V(G)$. Thus, $f(u) < \frac{m\ell}{2}$ for all $u \in V(G)$. Ergo, f is a persistent closed $\mathbb{Z}_{m\ell}$-valuation by Theorem 5.4.3.

■

There is no reason to believe that closed persistent valuations are limited to those described above. However, it may be that these are the most convenient ways of finding them.

Table 5.1 Examples of Fully Automorphic Decompositions

Prototype	Host	Restrictions	Reference
P_2	C_{2n}	None	
P_3	$C_{3n}(a, 2a)$	$3 \nmid a, 2a$	Proposition 2.2.3 [41]
K_3	$C_{3n}(a, b, a+b)$	$3 \nmid a, b, a+b$	Proposition 2.2.4 [62]

5.5 Fully Automorphic Divisors and Their Hosts

In the previous chapter, we introduced the idea of a fully automorphic G-decomposition of H as a means of determining which hosts would admit an automorphic decomposition. However, we gave no examples of such a decomposition. Using the notion of a cyclic decomposition and the results stated in Proposition 2.2.3 [41] and Proposition 2.2.4 [62], we give several examples in Table 5.1.

Since we are often restricted to circulants in our study of automorphic decompositions, the author has found it difficult to determine a fully automorphic decomposition. Other than the trivial cases (i.e., a cycle or a complete graph), the chromatic number of circulants has been understudied, except for the results of Heuberger [41] and Rosa et al. [62]. Those results suggest that the order of the host is a multiple of the order of the prototype in a fully automorphic decomposition. Thus, in studying fully automorphic divisors and their hosts, we encounter many of the same restrictions that were present in our discussion of closed persistent automorphic divisors. Namely, it is difficult to modify an existing valuation to ensure it decomposes the appropriate circulant. However, Proposition 2.2.5 does give a method of finding these decompositions. We only require that $n(G)|n(H)$ and that none of the differences are divisible by $n(G)$. This observation leads us to study p-modular valuations. A *(closed) p-modular $\mathbb{Z}_{pk}$-valuation* of G is a (closed) $\mathbb{Z}_{pk}$-valuation of G such that for all $uv \in E(G)$, $p \nmid |f(u) - f(v)|_{pk}$.

Theorem 5.5.1 *Let G be a graph of order p and take f to be a closed p-modular $\mathbb{Z}_{pk}$-valuation of G with $S = f^*(E(G))$. Then $H = C_{pk}(S)$ admits an automorphic G-decomposition and $\chi(H) = p$.*

Proof.

By Theorem 2.3.10, if f is a p-modular $\mathbb{Z}_{pk}$-valuation, it will induce a cyclic G-decomposition $\mathscr{D}$ of the circulant $H = C_{pk}(S)$. Proposition 2.2.5 implies that $\chi(H) \leq p$. If $\mathscr{D}$ is automorphic, we have that $\chi(H) \geq p$ by Theorem 3.2.1. Thus, it follows that $\chi(H) = p$. ■

This is an interesting result, as previously we have used the chromatic number to derive regularity results. However, now we are using an automorphic decomposition to determine the chromatic number. While this result does not give the chromatic number of all circulants, it does give the chromatic number of many circulant graphs where the colorability was unknown.

Proposition 5.5.2 *Let G be a graph and take $p \in \mathbb{Z}^+$.*

(i) If f is a p-modular $\mathbb{Z}_{pk}$-valuation on G, then we require that for all $uv \in E(G)$, $f(u) \neq f(v) \pmod p$.

(ii) If f is a closed p-modular $\mathbb{Z}_{pk}$-valuation on G, then we require that for all $u, v \in V(G)$, $f(u) \neq f(v) \pmod p$. As such, $n(G) \leq p$.

Proof.

(i) Let f be a p-modular valuation on G. Suppose to the contrary that there exists $uv \in E(G)$ such that $f(u) \equiv f(v) \pmod p$. This implies that the induced edge label, $|f(u)-f(v)|_{pk} \equiv 0 \pmod p$. Ergo, p divides $|f(u)-f(v)|_{pk}$, contrary to the definition of a p-modular valuation.

(ii) Let f be a closed p-modular valuation on G. Suppose to the contrary that there exist $u, v \in V(G)$ such that $f(u) \equiv f(v) \pmod p$. Since f is closed, there is an edge

$xy \in E(G)$ such that $|f(u) - f(v)|_{pk} = |f(x) - f(y)|_{pk}$. Computing the induced edge label on xy yields:

$$|f(x) - f(y)|_{pk} = |f(u) - f(v)|_{pk} \equiv 0 \pmod{p}.$$

Ergo, p divides $|f(x) - f(y)|_{pk}$, contrary to the definition of a p-modular valuation.

■

This gives us a method of constructing such a valuation.

Theorem 5.5.3 *The complete graph, K_p, has a closed p-modular $\mathbb{Z}_{pk}$-valuation for sufficiently large k.*

Proof.

It suffices to construct the required valuation. Let $V(K_p) = \{v_0, ..., v_{p-1}\}$ with $f(v_i) = a_i$. Suppose that $a_i < a_j$ for $i < j$. Further, assume that $a_i \equiv i \pmod{p}$. Note that if $a_{i+1} > 2a_i$ for all i, then the differences will all be distinct, provided the modulus is large enough. By choosing $pk > 2m$ where m is the largest edge label induced by f, we obtain the required order. Since all differences are distinct and each vertex label is from a different congruence class modulo p, we have constructed the required p-modular $\mathbb{Z}_{pk}$-valuation. Since any valuation on K_p is trivially closed, we have obtained a closed p-modular $\mathbb{Z}_{pk}$-valuation.

■

It should be noted that we do not need to choose the a_i to be from the congruence class of i modulo p. So long as we choose the elements to be from different congruence classes, we achieve the desired result. Further, we do not need to choose the vertex labels to be more than twice the previous label. This just gives us a convenient way of computing the labels. It is possible that the magnitude of the largest vertex label may be lowered dramatically by using a modified perfect difference set algorithm. Ignoring the restriction that the labels must come from successive congruence classes, we obtain the data in Table 5.2 via Theorem 5.5.3. Note that these valuations give examples of automorphic decompositions that are both fully automorphic and simple automorphic.

Table 5.2 p-Modular Valuations of K_p for $p \leq 6$

p	$f(V(K_p))$	$f^*(E(K_p))$	k
2	0,1	1	$k \geq 2$
3	0,1,5	1,4,5	$k \geq 5$
4	0,1,3,10	1,2,3,7,9,10	$k \geq 6$
5	0,1,3,7,19	1,2,3,4,6,7,12,16,18,19	$k \geq 8$
6	0,1,3,8,17,40	1,2,3,5,7,8,9,14,16,17,23,32,37,39,40	$k \geq 14$

We were able to extend the idea of a $\mathbb{Z}_n$-valuation from complete graphs to arbitrary graphs. We can use a similar technique to extend the idea of a p-modular valuation from complete graphs to arbitrary graphs.

Corollary 5.5.4 *All graphs of order p have a p-modular $\mathbb{Z}_{pk}$-valuation for suitably large k.*

Proof.

K_p has a p-modular $\mathbb{Z}_{pk}$-valuation by Theorem 5.5.3. We can obtain any graph of order p by deleting the appropriate edges from K_p. Since our definition of p-modular valuation places no restriction on pairs of non-adjacent vertices, this same valuation will also work for any arbitrary graph of order p. ∎

We can use the notion of graceful labellings to give additional examples of fully automorphic divisors.

Proposition 5.5.5 *Let G be a connected graph of order p, and let f a graceful labelling on G. f is a closed p-modular valuation on G if and only if G is a tree.*

Proof.

If G is not a tree, then $e(G) \geq n(G)$. Since f is a graceful labelling on G, then the edge labels induced by f are $\{1, 2, ..., e(G)\}$. Since $e(G) \geq n(G)$, it follows that there is an edge labelled $n(G) = p$, contrary to the definition of p-modular valuation.

If G is a tree, then $e(G) = n(G) - 1 = p - 1$. Since f is a graceful labelling on G, the edge labels induced by f are $\{1, 2, ..., p-1\}$. Since no edge is labelled with an element larger than $p - 1$, it follows that no edge is labelled with a multiple of p. By definition, f is a closed $\mathbb{Z}_{2p-1}$-valuation on G. Thus, f is a closed p-modular $\mathbb{Z}_{2p-1}$-valuation on G. ■

CHAPTER 6

EXTENSIONS

In this chapter, we briefly examine natural extensions of automorphic decompositions. In particular, we look at automorphic decompositions of multigraphs. We will also consider the possibility of decomposing H with respect to a family of prototypes.

6.1 Automorphic Decompositions of Multigraphs

In this section, we examine the possibility of extending the notion of an automorphic decomposition to the case where H and G are multigraphs. In order to do this, we first must define the G-decomposition graph in such a way that it will have multiple edges.

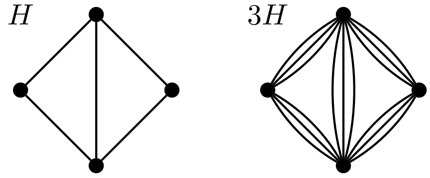
Definition 6.1.1 *Let $\mathscr{D}$ be a G-decomposition of H. The* multi-intersection graph *generated by $\mathscr{D}$, denoted $I(\mathscr{D})'$, has a vertex for each of the G-blocks of $\mathscr{D}$. Two vertices in $I(\mathscr{D})'$ share an edge of multiplicity k if and only if the corresponding G-blocks share k common nodes in H.*

Further, we may be interested in extending a simple graph H (or G) to give them multiple edges [39, 46]. Similarly, we may take a multigraph and remove the multiplicities from each edge to create a simple graph.

Definition 6.1.2 *If H is a multigraph, then the* simplification *of H, denoted $S(H)$, is the graph obtained from H by reducing all of the multiplicities to one. If H is a simple graph, then the* λ-extension of *H is the graph λH obtained from H by increasing the multiplicities of all the edges to λ (see Figure 6.1).*

Using these definitions, we may give several notions of automorphic decompositions of multigraphs.

Figure 6.1 An Extension of H



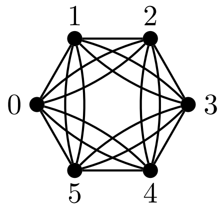
(i) If H and $I(\mathscr{D})$ are both simple and $I(\mathscr{D}) \cong H$, then we say that $\mathscr{D}$ is a *standard* automorphic decomposition of H. This case has been our primary concern in previous chapters.

(ii) If H is simple and $I(\mathscr{D})' \cong H$, we say that $\mathscr{D}$ is a *simple* automorphic decomposition of H. Note that this case requires that $\mathscr{D}$ is a simple decomposition. Hence, there is no conflict of terminology. Simple decompositions have been used as a means of obtaining a particular result. However, this case provides motivation for further study of such decompositions.

(iii) If H is a multigraph and $I(\mathscr{D}) \cong S(H)$, we say that $\mathscr{D}$ is a *substandard* automorphic decomposition. If $I(\mathscr{D})' \cong S(H)$, we say that $\mathscr{D}$ is a *subsimple* automorphic decomposition. Note that a subsimple automorphic decomposition requires that the decomposition is simple. If $I(\mathscr{D})' \cong H$, we say that $\mathscr{D}$ is a *true* automorphic decomposition. Note that if λ is the largest edge multiplicity in H, and H admits a true automorphic G-decomposition, then the decomposion must be λ-bounded. Thus we must have $n(G) \geq \lambda$.

Standard automorphic decompositions and simple decompositions have been mentioned, at least implicitly, throughout this dissertation. Thus, we will be considering the problem presented in the third case.

Figure 6.2 Multi-Circulant - $C_6((1,1),(2,2))$



The author has found that it is difficult to obtain an automorphic decomposition without using a cyclic decomposition. Unfortunately, a cyclic decomposition will not generally be simple, or even λ-uniform. However, there will be some regularity in the resulting multiplicities (see Proposition 2.3.16). Thus, we may obtain a solution to the third case by introducing a multi-circulant.

Definition 6.1.3 *Let $n \in \mathbb{Z}^+$ and $S \subseteq \mathbb{Z}_n^* \times \mathbb{Z}^+$ be given. Suppose that $S = \{(a_1, m_1), ..., (a_k, m_k)\}$. The* multi-circulant *$MC_n(S)$ is the undirected graph with vertex set $V = \mathbb{Z}_n$ and xy is an edge of multiplicity m_i if and only if $|x - y|_n = a_i$.*

An example of a multi-circulant is given in Figure 6.2.

It should be noted that if f is a closed $\mathbb{Z}_n$-valuation on G with $f^*(E(G)) = \{a_1, ..., a_k\}$, then we can obtain a substandard automorphic decomposition of $MC_n(S)$ by giving the edge of G labelled a_i multiplicity m_i. Similarly, if we know how many times various blocks intersect in the cyclic decomposition, then we can obtain a true automorphic decomposition of $MC_n(S)$ by adjusting the multiplicities of G in the same way. However, if we allow the use of multigraphs in this way, then we trivialize the problems introduced in the third case.

Further note, that if we allow multi-circulants, then we can weaken our notion of a valuation considerably. Suppose that f is a $\mathbb{Z}_n$-labelling of G such that for $xy \in E(G)$, $f^*(xy) = |x - y|_n$ is not injective. If f is such a labelling, we say that f is a *pseudo $\mathbb{Z}_n$-valuation* of G. For instance, suppose that f is a pseudo $\mathbb{Z}_n$-valuation of G and there are m_i edges of G labelled a_i. Then f will induce a cyclic G-decomposition of the multi-circulant $MC_n(S)$. If f is closed, then this decomposition will be a substandard automorphic decomposition. By choosing the multiplicities carefully, we can obtain a true automorphic decomposition. We expect that finding closed pseudo $\mathbb{Z}_n$-valuations is considerably easier than finding a closed $\mathbb{Z}_n$-valuation as there are fewer restrictions. Thus by including multi-circulants and pseudo valuations, the problem of finding automorphic divisors may become trivial.

For the reasons given above, we choose to avoid the notion of multi-circulants in the future. Instead, we focus on the notion of extensions of graphs. Let H and G be simple graphs and take $\mu, \lambda \in \mathbb{Z}^+$. Now, consider the problem of whether μG is an automorphic divisor of λH in any of the senses defined above. In particular, we are interested in whether μ and λ can be chosen in such a way that the divisibility requirement $e(\mu G)n(\lambda H) = e(\lambda H)$ from Theorem 3.2.1 is satisfied. In order to study this problem, we first list a couple of trivial properties of extensions.

Proposition 6.1.4 *Let H be a simple graph and let $\lambda \in \mathbb{N}$. We must have that $n(\lambda H) = n(H)$ and $e(\lambda H) = \lambda e(H)$.*

Proof.

In defining the λ-extension of H, we did nothing to alter the vertex set of H. Therefore, $n(\lambda H) = n(H)$. Further, we replaced each single edge of H with λ edges. Hence, $e(\lambda H) = \lambda e(H)$. ■

Note that for λH to admit an automorphic μG-decomposition, we must have that $\mu e(G)n(H) = \lambda e(H)$ by Theorem 3.2.1. Further note, if H admits an automorphic G-decomposition, then $e(G)n(H) = e(H)$. Thus, only the trivial solution of $\lambda = \mu$ will work. For this reason, we avoid cases where an automorphic decomposition is known, as well as cases where the divisibility requirements are met.

Supposing that $e(H) \neq n(H)e(G)$, is it always possible to choose μ and λ in such a way that the λH admits a subsimple or true automorphic μG-decomposition?

Note that if λH admits a subsimple automorphic μG-decomposition, then we require that the decomposition is simple. Hence, any two μG-blocks may share at most one common node in λH. Since any two μG-blocks may share more than one common node, it follows that if $uv \in E(\lambda H)$, then all of the edges between u and v in λH are partitioned by a single copy of μG. If $uv \in E(H)$, then there are λ such edges, and thus $\lambda = \mu$. As this is exactly the trivial case we were seeking to avoid, we will concentrate our attention on true automorphic decompositions.

Proposition 6.1.5 *If λH admits a true automorphic μG-decomposition, then we require the following:*

(i) $\chi(H) \geq n(G) \geq \lambda$.

(ii) $\lambda \geq \mu$.

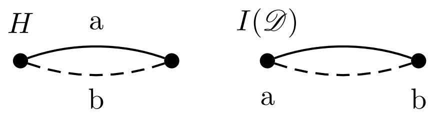
(iii) $\mu n(H)e(G) = \lambda e(H)$.

Proof.

(i) If λH admits a true automorphic μG-decomposition, then we require that this decomposition $\mathscr{D}$ be λ-uniform. In order for two μG-blocks to share λ common nodes in λH, we must have that $n(\mu G) = n(G) \geq \lambda$. Thus by Theorem 3.2.1, we must have that $\chi(H) \geq n(G) \geq \lambda$. Note that if $n(G) = \lambda$, then any two intersecting μG-blocks must share every node in λH. Thus any connected component of λH is isomorphic to an extension of the complete graph.

(ii) Assume to the contrary that $\mu > \lambda$. Since H and G are simple graphs and our decomposition is an edge partition, this implies that at least one edge is in multiple G-blocks. Hence a contradiction.

Figure 6.3 An Automorphic P_2-Decomposition of $2P_2$



(iii) By Theorem 3.2.1, we must have that $e(\lambda H) = n(\lambda H)e(\mu G)$. By Proposition 6.1.4, this is equivalent to $\lambda e(H) = \mu n(H)e(G)$.

■

It should be noted that there are no known examples of a λ-uniform automorphic decomposition, except when $\lambda = 1$, i.e., the simple case. Even then, we conjecture that the only simple automorphic divisor is $G \cong K_p$. As such, we expect that it would be difficult to find true automorphic decompositions.

Now, we examine various possible choices for λ and μ and look at their implications for H and G. In particular, if we choose λ and μ in such a way that the divisibility requirements are met, what other restrictions are imposed on H and G in order for μG to be an automorphic divisor of λH?

Case 1: Here we let $\mu = \ell e(H)$ and $\lambda = \ell n(H)e(G)$ for some $\ell \in \mathbb{Z}^+$. Note that $n(G) \geq \lambda = \ell n(H)e(G)$ by Proposition 6.1.5. If $n(H) \geq 3$, then it follows that $n(G) \geq 3\ell e(G)$. This would imply that G contains an isolated node by Proposition 1.1.6. However, if $n(H) \leq 2$, we must have that $H \cong P_2$ in order to avoid isolated nodes. Since we do not allow G to have isolated nodes and $n(G) \leq n(H)$, then we must have that $G \cong P_2$ as well. Thus we must have $\mu = \ell$ and $\lambda = 2\ell$. Since $n(G) \geq \lambda$ by Proposition 6.1.4, it follows that the only admissible choice is $\ell = 1$. The required decomposition is given in Figure 6.3.

Case 2: In our previous case, a triviality arose because we chose λ and μ too large. To rectify this, we choose our parameters to be smaller. Let $\mu = \frac{e(H)}{\ell e(G)}$ and $\lambda = \frac{n(H)}{\ell}$. Note that $\mu, \lambda \in \mathbb{Z}^+$. Thus if we restrict $\ell \in \mathbb{Z}^+$, we impose restrictions on G and H. For example,

$\ell e(G)|e(H)$ and thus $e(G)|e(H)$. Further, since $\ell e(G)|e(H)$, this implies that there exists $k \in \mathbb{N}$ such that $\ell = \frac{e(H)}{ke(G)}$. By Proposition 6.1.4, we have that:

$$n(G) \geq \lambda = \frac{n(H)}{\ell} = \frac{kn(H)e(G)}{e(H)}.$$

Note that if $n(H) = k\ell$, then the divisibility requirements are met and we have a contradiction. If $e(H) \leq n(H)$, then we must have $n(G) \geq ke(G)$. Since we do not allow isolated nodes, it follows that $k \leq 2$ by Proposition 1.1.6. If $k = 2$, then the only graph with $n(G) = 2e(G)$ and no isolated nodes is a disjoint union of P_2's [37]. Thus, we must have that $k = 1$ and $\ell = \frac{e(H)}{e(G)}$. This implies that $\lambda = \frac{n(H)e(G)}{e(H)}$ and $\mu = 1$. Since $n(G) \geq \lambda$, we must have that $\frac{n(H)}{e(H)} \geq 2$ in order for H to have no isolated nodes by Proposition 1.1.6. However, if $\frac{n(H)}{e(H)} \geq 3$, then H has an isolated node by Proposition 1.1.6. Thus, we must have that $\frac{n(H)}{e(H)} = 2$, and as such, H must be the disjoint union of P_2's. This reduces down to the trivial case.

Hence, we must have that $e(H) > n(H)$. This implies that $\ell > \frac{n(H)}{ke(G)}$, and thus, $\lambda < ke(G)$. If $n(G) \geq ke(G)$, then this reduces down to the above case. This implies the restriction $\lambda \leq n(G) < ke(G)$.

Case 3: Let $\mu = \frac{e(H)\ell}{n(H)}$ and $\lambda = e(G)\ell$. Since $n(G) \geq \lambda$, if $\ell \geq 3$, then G must have isolated nodes by Proposition 1.1.6. If $\ell = 2$ and equality holds, then G is the disjoint union of P_2's. If $\ell = 1$, then we have the restriction that $n(G) \geq e(G)$.

Case 4: $\lambda = \frac{e(G)\ell}{e(H)}$, $\mu = \frac{\ell}{n(H)}$. This implies that $\ell = n(H)\mu$. Further, we must have that $\lambda = \frac{e(G)n(H)\mu}{e(H)}$. It follows that $e(H)|e(G)n(H)\mu$. Note $n(G) \geq \lambda = \frac{e(G)n(H)\mu}{e(H)}$. If $\frac{n(H)\mu}{e(H)} \geq 3$, then G contains isolated nodes by Proposition 1.1.6. If $\frac{n(H)\mu}{e(H)} = 2$ and equality holds, then G is the disjoint union of P_2's. Thus we must have $\frac{n(H)\mu}{e(G)} = 1$. This is equivalent to $\mu = \frac{e(H)}{n(H)}$, i.e., $n(H)|e(H)$. Further, this implies that $\lambda = e(G)$ and as such $n(G) \geq e(G)$. Moreover, H is a disjoint union of complete graphs. If $n(G) > e(G)$, then either G is disconnected or G is a tree by Proposition 1.1.6.

Case 5: $\lambda = \frac{\ell}{e(H)}$ and $\mu = \frac{\ell}{n(H)e(G)}$. Note that for $\ell \in \mathbb{N}$, we must have $e(H)|\ell$ and $n(H)e(G)|\ell$. Let $e(H) = ka$ and $n(H)e(G) = kb$ where $k, a, b \in \mathbb{N}$ and $gcd(a, b) = 1$. Note that $k = gcd(e(H), n(H)e(G))$. So, $\ell = kab$, $b = \frac{n(H)e(G)}{k}$, and $a = \frac{e(H)}{k}$. Thus, $\lambda = b$ and $\mu = a$ and as such $n(G) \geq b \geq a$. This implies that $n(G) \geq \frac{n(H)e(G)}{k}$. If $n(H) \geq 3k$, then G has isolated nodes by Proposition 1.1.6. If $n(H) = 2k$ and equality holds, then G is the disjoint union of P_2's. So we may assume that $n(H) = k$ and thus $\lambda = e(G)$. If $n(G) = \lambda$, then H is a disjoint union of complete graphs. If $n(G) > e(G)$, then G is either disconnected or a tree.

Case 6: $\lambda = \frac{n(H)\ell}{e(H)}$, $\mu = \frac{\ell}{e(G)}$. This implies $e(H)|n(H)\ell$ and $e(G)|\ell$. Thus $\lambda = \frac{n(H)e(G)\mu}{e(H)}$. As such, this case reduces down to Case 4.

Note that in many of these "easy" choices, we are left with the possibility that H is a disjoint union of complete graphs. This case is summarized in Heinrich [39]. Another possibility for multigraphs is that G is the disjoint union of P_2's, a trivial case. Thus, the only interesting possibility is where $n(G) \leq e(G)$. We leave this problem open for future research.

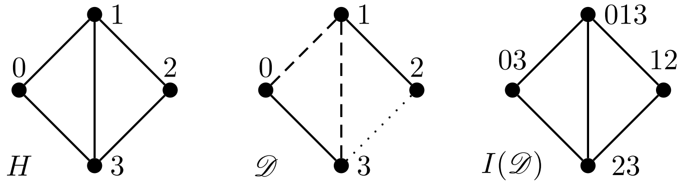
6.2 Families of Prototypes

It is common in the literature [7, 9, 44, 47] to examine the possibility of decomposing the host graph, H, with respect to a family of prototypes, $\mathcal{K}$. If we are able to partition the edge set of H such that each part of the partition, $\mathscr{D}$, is isomorphic to an element of $\mathcal{K}$, then we say that $\mathscr{D}$ is a *$\mathcal{K}$-decomposition of H*. Intersection graphs generated by a $\mathcal{K}$-decomposition and automorphic $\mathcal{K}$-decompositions are defined analogously.

Previously, we have avoided this case. We believe that the necessary conditions for the existence of such a decomposition are nearly trivial. As such, we suspect that the existence of an automorphic $\mathcal{K}$-decomposition is uninteresting, even for graphs that previously could not host an automorphic decomposition.

Example 6.2.1 *Let H be the graph obtained from K_4 by deleting a single edge. Suppose the nodes of H are identified as shown in the left of Figure 6.4. Let $\mathcal{K} = \{P_2, P_3\}$. Let $\mathscr{D}$ consist of three P_2-blocks,* 03, 12, *and* 23, *as well as a single P_3-block,* 013. *Note that this is clearly a decomposition of H, as shown in the center of Figure 6.4. Further note that* 013 *and* 23

Figure 6.4 An Automorphic $\mathcal{K}$-Decomposition of $K_4 - e$



intersect every other block. However, 12 *and* 03 *are disjoint. Thus,* $I(\mathscr{D}) \cong H$ *as shown in the right of Figure 6.4. We note that* H *cannot host an automorphic* G*-decomposition for any choice of* G *because it fails to meet the divisibility requirements of Theorem 3.2.1.*

There are many graphs that cannot host an automorphic decomposition, even when we allow for decomposition by a family of prototypes.

Theorem 6.2.2 *Let* H *be a graph such that* $n(H) \geq e(H)$*.* H *admits an automorphic* $\mathcal{K}$*-decomposition if and only if* H *is isomorphic to a disjoint union of cycles.*

Proof.

If H is a disjoint union of cycles, it admits an automorphic P_2-decomposition by Theorem 3.2.4.

Conversely, assume that H admits an automorphic $\mathcal{K}$-decomposition $\mathscr{D}$. Note that the number of blocks in a $\mathcal{K}$-decomposition is at most $e(H)$. Since the order of a $\mathcal{K}$-decomposition graph is equal to the number of blocks in the decomposition, it follows that $n(I(\mathscr{D})) \geq e(H)$ for any choice of $\mathcal{K}$ and $\mathscr{D}$. If $\mathscr{D}$ is to be automorphic, we must have that $n(I(\mathscr{D})) = n(H)$ by Theorem 3.2.1. However, if $n(H) > e(H)$, then we have:

$$e(H) \geq n(I(\mathscr{D})) = n(H) > e(H).$$

Since $e(H)$ cannot be strictly larger than itself, we have a contradiction. Thus we must have $n(H) = e(H)$. This implies that the number of blocks is precisely the edge count.

However, for this to be possible, our only admissible prototype is P_2. Note that H admits an automorphic P_2-decomposition if and only if H is a disjoint union of cycles by Theorem 3.2.4. ■

CHAPTER 7

CONCLUDING REMARKS

The study of automorphic decompositions is related to several classic problems in graph theory and combinatorics. The first is Ringel's Conjecture [65], which would imply that any tree is an automorphic divisor of a complete graph. In an attempt to prove this, Rosa [66] introduced β-valuations which would induce a cyclic decomposition of the complete graph. Rosa's β-valuations were popularized by Golomb [32] who called such labellings graceful. A tremendous amount of effort has been spent trying to prove Rosa's Graceful Tree Conjecture [31, 66]. We have introduced new valuations that generalize Rosa's valuations from complete graphs to circulants. We were later able to generalize this further by allowing the labels on a graph to come from arbitrary groups. This allowed us to construct new divisors from old ones using subproducts and subjoins.

While cyclic, or more generally Γ-translational, decompositions are often used to algebraically construct a decomposition, these can also be obtained by the existence of a design. In particular, the existence of a projective plane of order n is equivalent to the existence of an automorphic K_{n+1}-decomposition of K_{n^2+n+1}. As such, the problem of finding automorphic decompositions is related to the Prime Power Conjecture as well as the Bruck-Ryser-Chowla Theorem. In turn this is related to finding perfect difference sets and Golomb rulers.

In this dissertation, we listed many necessary conditions for the existence of an automorphic decomposition in Theorem 3.2.1. Several of these are in the form of Linear Diophantine Equations [38]. As such, our problem is related to the problem of Frobenius [21, 25, 54, 70] as well as integer programming [67]. These necessary conditions allowed us to show that many graphs cannot host an automorphic decomposition.

We conjecture that only even regular graphs can host an automorphic decomposition. We were able to show that a graph hosts an automorphic P_2-decomposition if and only if it is a disjoint union of cycles in Theorem 3.2.4. By assuming that any two blocks share at most one common node (i.e., a simple decomposition), we were able to show that if G is a d-regular graph of order p and H hosts a simple automorphic G-decomposition, then H is $p(p-1)$-regular, and as such $G \cong K_p$. If we instead assume that $\chi(H) = n(G)$ (i.e., a fully automorphic decomposition), we obtain regularity results for a larger class of prototypes. However, the general theorem remains open.

We also have conjectured that any graph is an automorphic divisor. Because of the wide variety of graphs that can be Γ-valuated, the author has found it difficult to show otherwise. We suspect that the difficulty of proving this conjecture is roughly equivalent to proving Ringel's Conjecture. However, we were able to extend the known classes of graphs that are automorphic divisors. We also examined the problem of determining a persistent automorphic divisor as well as that of determining a fully automorphic divisor. As our examples are often based on cyclic decompositions, we need the appropriate valuation to induce the decomposition. As closed valuations are often only valid for a single group, the author has found this to be a difficult problem.

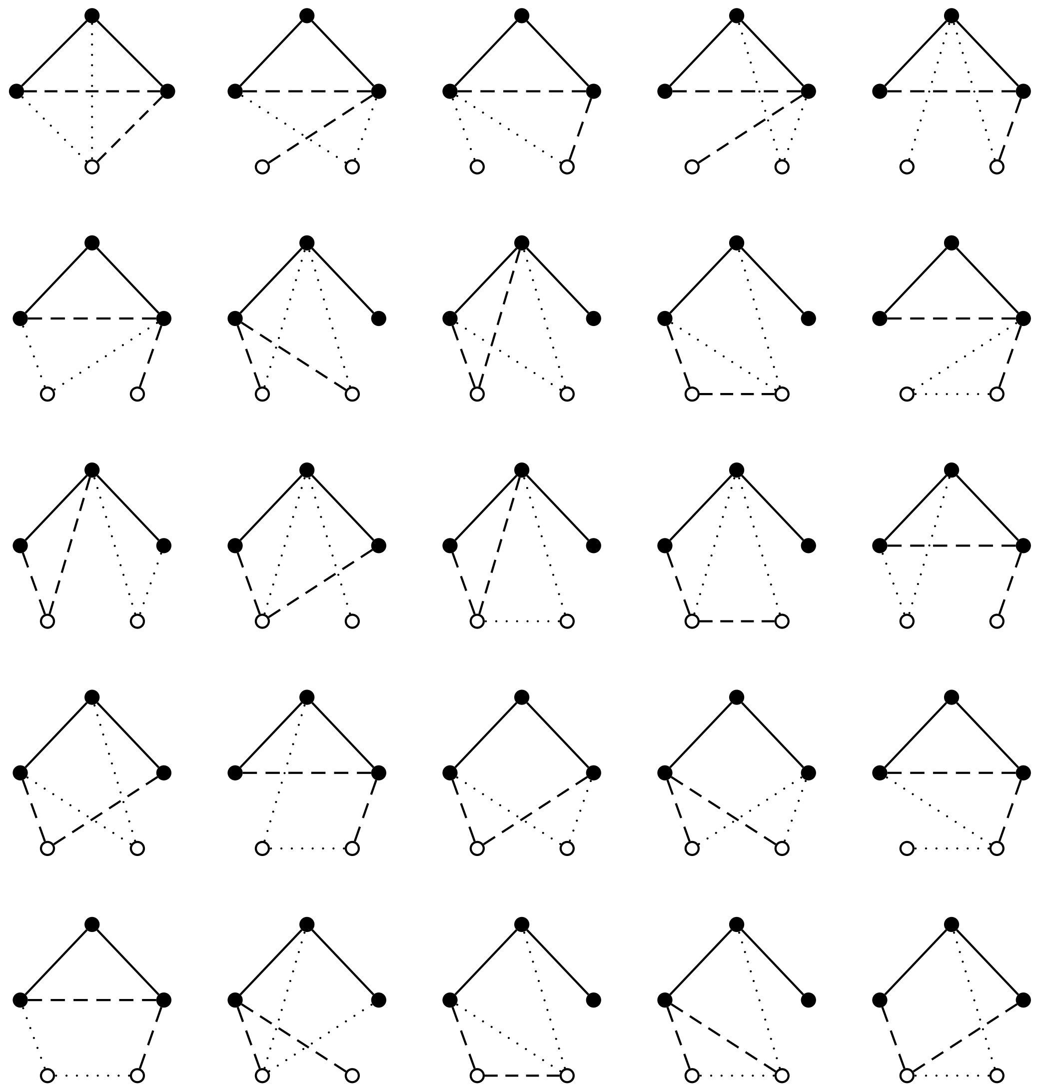
In closing, we believe that the study of automorphic decompositions warrants future research. We hope that this dissertation will serve as a resource for those who wish to further pursue this area.

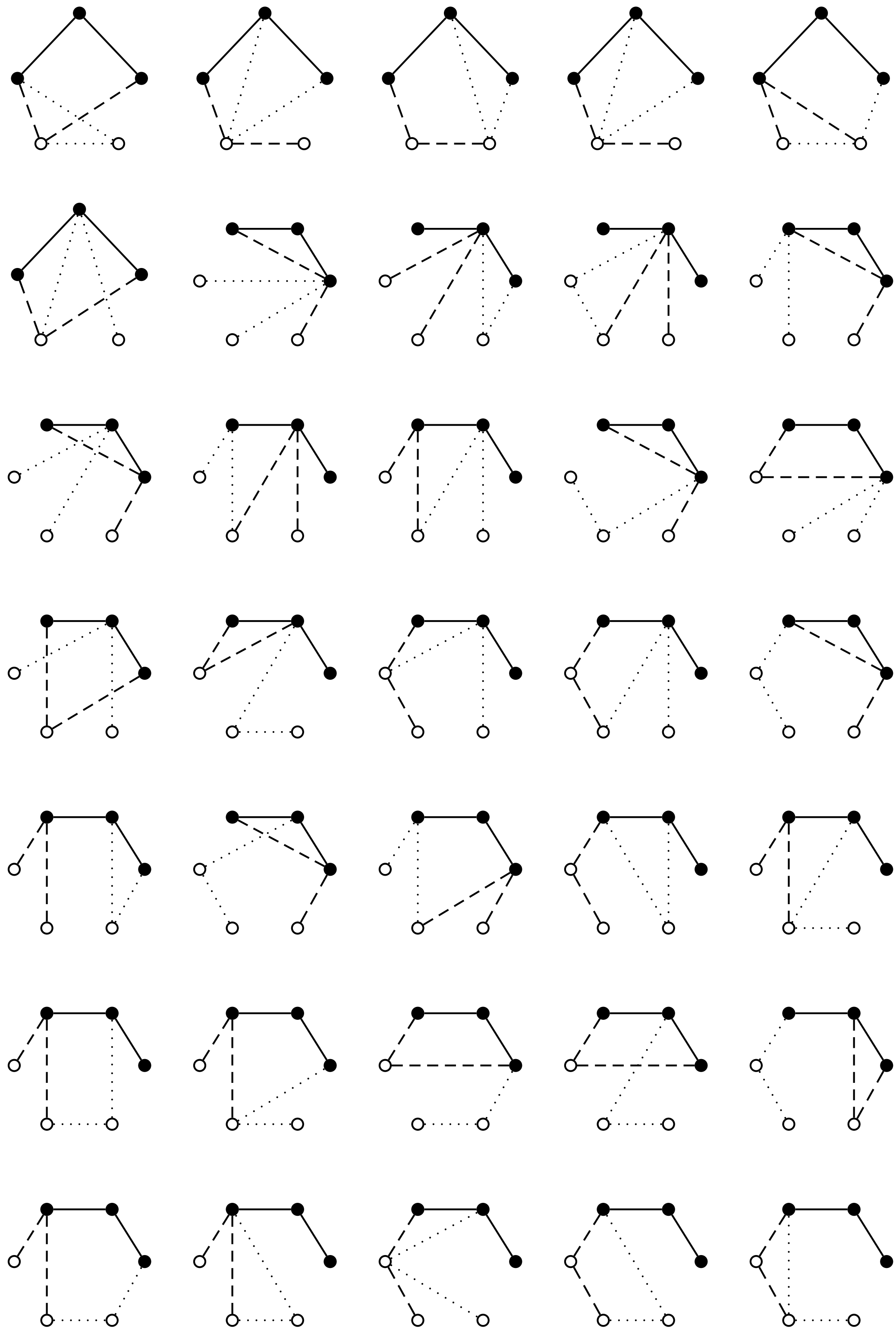
APPENDICES

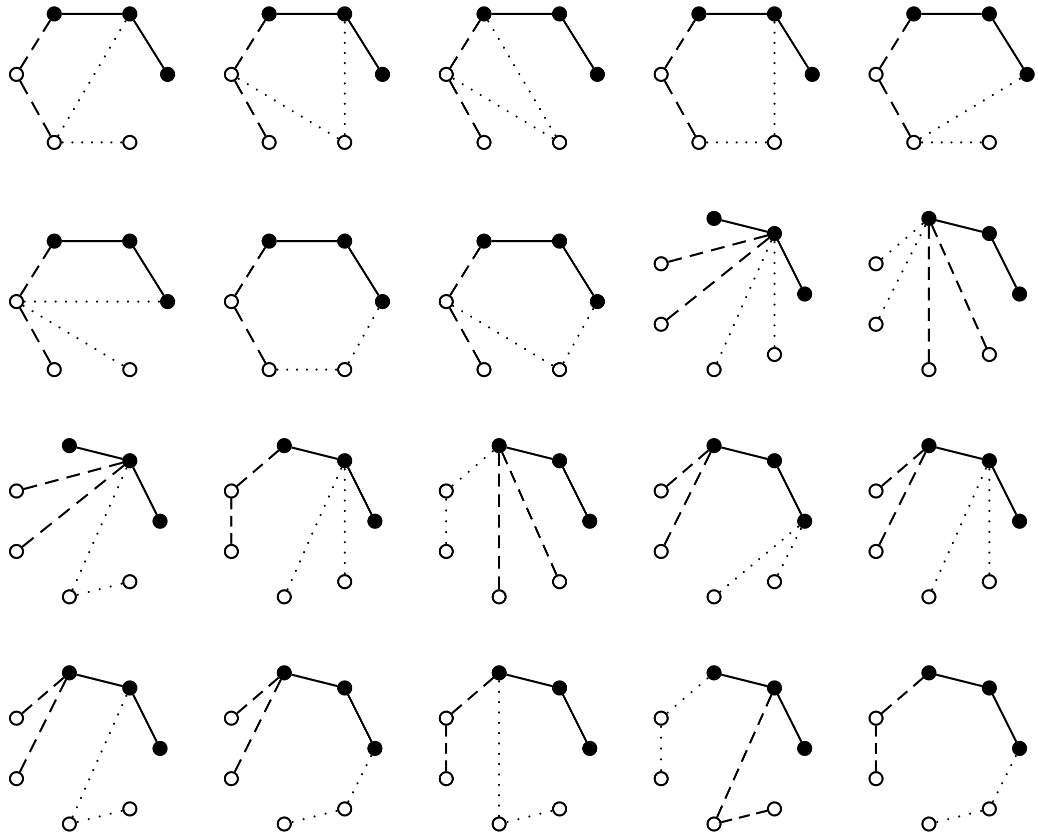
Appendix A

List of P_3 Intersections Representing a Vertex of Degree Two

The blocks with solid lines and nodes represent a vertex of degree two. The blocks with dashed or dotted lines and hollow nodes represent the intersecting blocks.







It should be noted that in sixty-nine of these, all of the blocks mutually intersect. Thus, if we know that the vertex of degree two is not part of a triangle, we can eliminate most of these intersections. We can also trim this list by restricting ourselves to a simple decomposition.

Appendix B

List of Notation

$V(G)$ - the vertex (node) set of a graph G

$E(G)$ - the edge set of a graph G

$n(G)$ - the order of a graph G

$e(G)$ - the size of a graph G

$deg_G(v)$ - the degree of a vertex v in the graph G

$dseq(G)$ - the degree sequence of G

$\delta(G)$ - the minimum degree of G

$\bar{d}(G)$ - the average degree of G

$\Delta(G)$ - the maximum degree of G

$\bar{G}$ - the complement of G

$\omega(G)$ - the clique number of G

$\alpha(G)$ - the independence number of G

$\chi(G)$ - the chromatic number of G

P_n - a path on n vertices

C_n - a cycle on n vertices

K_n - a complete graph on n vertices

$K_{n,m}$ - a complete bipartite graph, with independent sets of size n and m

Q_n - an n-dimensional hypercube

$C_n(S)$ - a circulant of order n and difference set S

$C(a_1, ..., a_n)$ - a Caterpillar

G_A - the subgraph of G induced by $A \subseteq V(G)$

$G \cong H$ - G is isomorphic to H

$\mathbb{N}$ - the set of non-negative integers

$\mathbb{Z}$ - the set of integers

$\mathbb{Z}^+$ - the set of positive integers

$\mathbb{Z}_n$ - the integers modulo n

$\mathbb{Z}_n^+$ - the set $\{x \in \mathbb{Z}_n : 0 < x < n/2\}$

$\mathbb{Z}_n^*$ - the set $\{x \in \mathbb{Z}_n : 0 < x \leq n/2\}$

$G \cup H$ - the disjoint union of G and H

$G \vee H$ - the join of G and H

$G \Box H$ - the Cartesian product of G and H

$G \times H$ - the direct product of G and H

$G \boxtimes H$ - the strong product of G and H

$f(V(G))$ - the set of vertex labels from a function f

$f^*(E(G))$ - the set of edge labels induced by a labelling f on G

$f^*(G)$ - the set of pairwise vertex labels of G induced by f

$G|H$ - G is a divisor of H

$G\|H$ - G is a balanced divisor of H

$I(\mathscr{D})$ - the intersection graph generated by a decomposition $\mathscr{D}$

$I(\mathscr{D})'$ - the multi-intersection graph generated by a decomposition $\mathscr{D}$

$S(H)$ - the simplification of a multigraph H

λH - the λ extension of a simple graph H

λ_{xy} - the covalency of x and y

$gcd(S)$ - the greatest common divisor of a set of positive integers S

$L(H)$ - the line graph of H

$o(\Gamma)$ - the order of a group Γ

$i(\Gamma)$ - the set of involutions in a group Γ

$C_\Gamma(S)$ - a Γ-transulant with difference set S

$MC_n(S)$ - a multi-circulant with difference set S

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