AN INVESTIGATION OF PROPORTIONAL THINKING AMONG HIGH SCHOOL STUDENTS

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AN INVESTIGATION OF PROPORTIONAL THINKING
AMONG HIGH SCHOOL STUDENTS

A Dissertation
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy
Curriculum and Instruction

by
Kirsten Bernasconi Dooley
December 2006

Accepted by:
Dr. Robert M. Horton, Committee Chair
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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS SUGGESTS THAT FORMAL PROPORTIONAL REASONING INSTRUCTION TAKE PLACE DURING GRADES FIVE THROUGH EIGHT (1989). WHILE EXTENSIVE RESEARCH HAS EXPLORED BOTH PRE-INSTRUCTIONAL AND INTRA-INSTRUCTIONAL STUDENT PROPORTIONAL THINKING STRATEGIES, LITTLE RESEARCH HAS BEEN DONE ON POST-INSTRUCTIONAL PROPORTIONAL THINKING STRATEGIES USED BY HIGH-SCHOOL AGED STUDENTS. AS PROPORTIONAL REASONING IS FUNDAMENTAL FOR SUCCESS IN ALGEBRA AND OTHER ADVANCED MATHEMATICS, IT SEEMS RELEVANT TO VERIFY THAT THESE STUDENTS RETAIN THEIR REASONING ABILITIES AND CAN APPLY THEM IN BOTH HIGH SCHOOL AND COLLEGE MATHEMATICAL SITUATIONS.

THIS STUDY Sought TO: (1) DETERMINE WHETHER HIGH-SCHOOL STUDENTS FROM SPECIFIC RURAL, LOW-INCOME SCHOOL ENVIRONMENTS POSSESSED ADVANCED PROPORTIONAL REASONING ABILITIES, (2) EXPLORE STUDENTS’ CONCEPTUAL UNDERSTANDING OF THE “CROSS MULTIPLY AND DIVIDE ALGORITHM,” AND (3) EVALUATE THE IMPACT OF MANIPULATIVES ON STUDENT THINKING. THIS SYSTEMATIC INVESTIGATION OF THE PARTICIPANTS’ PROPORTIONAL REASONING ABILITIES WAS COMPRISED PRIMARILY OF INTERVIEWS AND OBSERVATIONS.

ONE HUNDRED AND SEVEN HIGH SCHOOL STUDENTS WERE INVOLVED IN THE STUDY, TWENTY-ONE OF WHOM PARTICIPATED IN ONE-ON-ONE IN-DEPTH INTERVIEWS. THE RESEARCH FOUND: (1) ONLY TWO OF THE TWENTY-ONE INTERVIEWEES (9.5%) EXHIBITED ADVANCED PROPORTIONAL REASONING SKILLS, (2) NINETEEN OF THE INTERVIEWEES (90.5%) WERE UNABLE TO APPLY THEIR KNOWLEDGE OF PROPORTIONS AND THE “CROSS MULTIPLY AND DIVIDE” ALGORITHM TO SOLVE NON-TRADITIONAL PROBLEMS, AND (3) EVIDENCE INDICATED THAT USING MANIPULATIVES HELPED STUDENTS MOVE TOWARD SOLUTIONS TO THE PROPORTIONAL REASONING PROBLEMS.
ACKNOWLEDGMENTS

This project could not have been completed alone and I am indebted to many. I would like to thank Dr. Horton for his steady guidance and my committee members for their constructive comments. I appreciate the support and friendship of Hope Marchionda who was with me every step of the way. I’m grateful to my family for their encouragement when my belief was lacking, but mostly, for their unconditional love. Finally, I’d like to thank my husband for simply standing by my side.
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CHAPTER 1
INTRODUCTION

Introduction

Proportional reasoning is one of the building blocks of mathematical thinking. According to the National Council of Teachers of Mathematics (NCTM) (1989) “the ability to reason proportionally develops in students through grades 5-8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development” (p. 82).

This chapter begins with a discussion on proportional reasoning and its importance. The next section of this chapter addresses the inequities of educational opportunities in low socioeconomic areas. The chapter continues with a brief discussion of the gaps in the current research and concludes with the statement of the research questions.

Proportional Reasoning

Ben-Chaim, Fey, Fitzgerald, Benedetto, and Miller (1998) stated that proportional reasoning “involves mathematical relationships which are multiplicative in nature” (p. 249). All proportional reasoning problems involve co-variation between two variables (Spinillo & Bryant, 1999). Lesh, Post, and Behr (1988) claimed that the “essential characteristics of proportional reasoning…involve reasoning about the holistic relationship between two rational expressions such as rates, ratios, quotients, and fractions” (p. 93). Some examples of proportional reasoning problems include:
• Your local pizza shop is having a special of two large one-topping pizzas for $15.00. How many whole pizzas can you buy for $60.00?

• You are in charge of providing refreshments for the end-of-the-summer party. You need to make lemonade for all 50 students and each student will drink approximately 2 cups of lemonade. The lemonade mix package states that for every one cup of liquid lemonade mix used, three cups of water are required. How many cups of lemonade mix are necessary?

• For every 50 people who attend the school fair, approximately 37 of them will purchase a raffle ticket in addition to paying the entrance fee. After buying the prizes for the raffle, the school makes a profit of $1.25 for each raffle ticket sold. Seven hundred twenty-three people have purchased tickets to the fair. How much money can the school expect to make on the raffle?

• Dan and Tasha both left home at 10:00 am. Dan walked 2 miles to the post office, 3 miles from the post office to the zoo, and 1 mile from the zoo to home. Tasha walked 2.5 miles to her friend’s house, 1.5 miles from her friend’s house to the drug store, 3 miles from the drug store to home. Both Dan and Tasha arrived home at exactly 12:30 pm. Describe exactly how you can tell who walked faster, Dan or Tasha (Lamon, 1999)?

• You had a picture on your computer and you made it \( \frac{3}{4} \) (or 75%) of its original size. You changed your mind and you want it back to its original size again. What fraction of its present size should you tell the computer to make it in order to restore its original size (Lamon, 1999)?
The Flavorful Fruit Juice Company bottles various fruit juices. A small barrel of apple juice is mixed using 12 cans of apple concentrate and 30 cans of water. A large barrel of raspberry juice is mixed using 16 cans of raspberry concentrate and 36 cans of water. Which barrel will be fruitier (Lamon, 1999)?

As evident from above, proportional reasoning takes many different forms. There is no formula or predetermined set of steps used to solve every proportional-type problem. Solutions involve the use of reasoning skills.

The Importance of Proportional Reasoning

Practical Purpose

Proportional reasoning concepts are used in everyday life. For example, consider a conversation I had with a colleague whom I will refer to as Jennifer.

Jennifer (J): I brought you back a candy bar from Epcot Center.

Kirsten (K): Oh, thanks. How many calories does it have?

J: I am not sure. I was trying to figure it out. They list the nutritional information in kilojoules. I’m trying to convert it now.

K: I don’t know how many kilojoules are in a calorie.

J: I looked it up on the Internet, but I still can’t do it. I always forget how you’re supposed to convert. You’re good at math; can you just do it for me?

Jennifer lacked proportional reasoning abilities. She tried to recollect an algorithm to use for the conversion rather than thinking through the problem. If proportional reasoning is not understood conceptually, but algorithmically, it becomes more difficult to transfer and use in “real-life.” Chapin and Anderson (2003) stated that helping students to develop a robust understanding of ratios, rates, and proportions and when to
appropriately use them is not an easy task. Teachers use a variety of strategies to aid students by “creating classroom situations that extend students’ knowledge and help them recognize the same mathematical ‘structure’ in different contexts” (p. 421).

**Academic Purpose**

The National Council of Teachers of Mathematics (NCTM) (2000) identified the following topics requiring proportional reasoning abilities in the high school curriculum:

1. Linear equations
2. Rates
3. Rational numbers and expressions
4. Similar figures
5. Area and volume relationships.

Other researchers have also stated the importance of proportional thinking for development in a variety of areas. Slovin (2000) stated “the study of ratio and proportion lays a foundation for first-year algebra; and subsequent mathematics and science study assumes that students can reason proportionally” (p. 58). Further, these reasoning skills challenge students to think and develop and help “expand structures necessary for continued intellectual development” (Behr, Lesh, Post, & Silver, 1983, p. 91). Spinillo and Bryant (1999) asserted that proportions are important in learning advanced mathematics, but, also, are crucial in dealing with basic scientific concepts, such as temperature and density. Johnson (2000) elaborated that

A full understanding of proportionality opens the doors to many interesting applications of mathematics. A student who understands the proportionality involved in the areas and volumes of similar solids, for example can understand
why elephants and hippopotamuses have such short massive legs; why a mouse can fall from a height of several stories and walk away, whereas a human will usually suffer fatal injuries (p. 102).

Failing to develop proportional reasoning abilities yields dire consequences. Lamon (1999) stated

The losses that occur because of the gaps in conceptual understanding about fractions, ratios, and rational numbers are incalculable. The consequences of ‘doing,’ rather than understanding, directly or indirectly affect a person’s attitudes toward mathematics, enjoyment and motivation in learning, course selection in mathematics and science, achievement, career flexibility, and even the ability to fully appreciate some of the simplest phenomena in everyday life (p. xi).

Inequities of Educational Opportunities

The National Assessment for Educational Progress (NAEP) is a nationwide assessment of students’ knowledge in various subject areas. Students in grades 4, 8, and 12 are tested. The test in mathematics consists of multiple-choice and constructed-response questions and was administered and analyzed, most recently, in 2005, 2003, and 2000. The survey gathers student information such as household income and parents' level of education. Student responses to mathematics questions are sorted into one of four groups: below basic, basic, proficient, and advanced. The 2005 report from the National Assessment for Educational Progress (NAEP) (http://nces.ed.gov/nationsreportcard/mathematics) asserted that students who are eligible for free/reduced - price lunch scored, on average, lower than those who were not eligible. The report further suggested that students who reported higher levels of parent education
scored better on the test. There was a statistically significant gap, across all three grade bands, between the performances of White and Asian/Pacific Islander students and the performances of Black, Hispanic, and American Indian/Alaska Native students. This gap may be due, in part, to the fact that “poor and non-white students have less-qualified teachers, take less-challenging classes and attend schools with less funding than their white and affluent peers” (Rubin, 2004, p. 6). Teachers in poor schools also are less likely to have specific content knowledge. Many teachers in those schools are not certified and less than half have bachelor degrees in their particular fields (Rubin, 2004). These students are also given fewer opportunities to develop proportional reasoning abilities. Karplus, Pulos, and Stage (1983) demonstrated this fact when they discovered that “though academically upper-track or upper middle-class students used proportional reasoning increasingly after about age 12 years, only a small fraction of urban low-income and academically lower-track students used proportions at age 14 or even 17 years” (p. 46).

**Missing Research: Older Students**

Proportional reasoning has been called the capstone of elementary mathematics and the cornerstone of high school mathematics (Lesh, Post, & Behr, 1988). The NCTM suggested that formal proportional reasoning instruction should take place during grades 5 through 8 (1989). Over the past twenty years, extensive research has been completed measuring both preinstructional proportional reasoning student strategies (Lamon, 1993a; Resnick & Singer, 1993; Lamon, 1994; Spinillo & Bryant, 1999; Spinillo, 2002) and during-instruction student strategies (Karplus, Pulos, & Stage, 1983; Kaput & West,
Little research has been done on post-instructional high-school aged students. As proportional reasoning is fundamental for success in Algebra and more advanced mathematics, it seems relevant to verify that these students do indeed retain their reasoning abilities so that they can apply those abilities in high school and college classrooms. The few studies conducted on older students focused on error analysis rather than assessing the students’ abilities (Hart, 1984). Karplus studied proportional reasoning of older students in the 1970’s, but those studies did not try to generalize student knowledge, but reported results from particular tasks (Karplus, Karplus, & Wollman, 1974). Another study, done in Malaysia, assessed the proportional reasoning abilities of ninth grade students (Singh, 2000). A gap exists in the research literature regarding proportional reasoning abilities of high school aged students.

Research Questions

Proportional reasoning abilities are important for everyone from late elementary school through adulthood. They aid in both formal classroom participation and in “real-life” situations. Unfortunately, the research (Rubin, 2004) suggests some students, particularly those residing in low socioeconomic areas, may be less likely to acquire and use these valuable tools. Are high school students in poor school districts able to reason proportionally? As there is no research addressing that question, this study focuses on that question.
The population of students participating in this study comes from the poorest counties in a particular southeastern state where many students do not complete high school. My research seeks to answer the following questions:

1. Do students from rural, low-income school environments who have completed Algebra I (and Geometry) have advanced proportional reasoning skills?
2. Can students from rural, low-income school environments who demonstrate an ability to solve proportions algorithmically solve problems that do not fit into the cross multiply and divide mold (do they understand, conceptually, the mathematics behind the algorithm)?
3. Can high school students from rural, low-income school environments model proportional problems using concrete materials? If so, how does that modeling influence their thinking?

This systematic investigation of the participants’ proportional reasoning abilities was comprised primarily of interviews and observations.
CHAPTER 2
LITERATURE REVIEW

Introduction

“The domain [proportional reasoning] represents a critical juncture at which many
types of mathematical knowledge are called into play and a point beyond which a
student’s understanding in the mathematical sciences will be greatly hampered if the
conceptual coordination of all the contributing domains is not attained” (Lamon, 1994, p. 90).

In this chapter, literature relevant to students’ proportional reasoning understanding is discussed. The first section contains a description of proportional reasoning and its necessary prerequisite concepts. A description of students’ informal and preproportional solution strategies is contained in the second section of this chapter. Finally, Chapter 2 concludes with an identification of different semantic structures of proportional reasoning. This final section also contains methods used to analyze student proportional thinking.

Understanding Proportional Reasoning

*Proportional Reasoning*

If a chicken and a half lays an egg and a half in a day and a half, how many eggs
do five chickens lay in six days (Karplus, Pulos, Stage, 1983)? On the surface this
question might appear difficult, silly, or both. To answer the question, a strong sense of
proportional reasoning is needed. Despite the perception of the question, proportional
reasoning is necessary when facing problems dealing with equivalence, variables, and
transformations (Lesh, Post, & Behr, 1988). Lesh, Post, and Behr (1988) stated that proportional reasoning is “a form of mathematic reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought” (p. 93). Considered both a capstone of elementary mathematics and a cornerstone of all advanced mathematics, proportional reasoning is a pivotal concept (Lesh, Post, & Behr, 1988). Students must have substantial prerequisite understanding of certain ideas to develop a significant ability to reason proportionally.

Prerequisite Concepts

It is valuable for educators to identify the necessary mechanisms needed to engage in proportional thinking and to verify that students possess that prerequisite knowledge. Lamon supported this concept and asserted that “by the time a child engages in proportional reasoning, prior knowledge and experience are so extensive and interactive that to ask how that student arrived at any given point in his or her learning is not to ask the important question” (1994, p. 90).

Absolute versus Relative Thinking

The concept of change is fundamental in a child’s early mathematical career. Most children first learn to think absolutely (Lamon, 1999). Lamon (1999) offered the following example. Suppose that we measure the height of two trees; the oak is eight feet tall and the evergreen is fourteen feet tall. Which tree is taller and by how much? The answer is, of course, the evergreen is taller by 6 feet. Now suppose we measure the same trees again in five years and the oak is ten feet tall and the evergreen is sixteen feet tall.
Which tree grew more over the past five years? This question can be answered in one of two ways. In absolute terms both trees grew the same amount, two feet. In relative terms the oak tree grew 2 feet or \( \frac{2}{8} \) of its original height. The evergreen grew 2 feet or \( \frac{2}{14} \) of its original height. In relative terms, the oak tree grew more because it increased its height by \( \frac{2}{8} \) (or \( \frac{1}{4} \) or 25%) while the evergreen increased its height by \( \frac{2}{14} \) (or \( \frac{1}{7} \) or 14.29%).

The key to relative thinking is comparing one measurement to itself.

Another example of absolute versus relative thinking is evident within the context of store sale prices. Absolute change would occur if the sale was $5.00 off of a particular item and relative change would occur if the sale was 10% off. The absolute relationship is additive while the relative relationship is multiplicative (Langrall & Swafford, 2000). The development of reasoning used to make sense of multiplicative relationships is important. Lamon (1999) expounded on this idea and asserted that relative reasoning is crucial in the following proportional concepts:

1. the relationship between the size of pieces and the number of pieces
2. the need to compare fractions relative to the same unit
3. the meaning of a fractional number
4. the relationship between equivalent fractions
5. the relationship between equivalent fraction representations (p. 13).

Unitizing

If asked to draw a pictorial representation of the fraction \( \frac{3}{4} \), most people would draw a circle divided into four equal pieces with three of those pieces shaded. In the previous scenario, the “whole” would be the entire circle. This representation works for
simplistic, beginning fractional understanding, but to develop more sophisticated mathematical ideas, students must be able to construct different sized units and reinterpret situations in terms of that defined unit (Lamon, 1994). For example, imagine a classroom with 25 students. Suppose 5 groups were formed, each containing five students. One way of describing the number of students in the class would be 25 students. Another way would be to say there are five groups of five students. In the second scenario, the unit is now composite, consisting of a group rather than an individual. Further, suppose that 2 of the five groups finished a project. Now, we could say that 10 of the 25 students were done (using an individual student as the unit) or we could say that 2 of the 5 groups (using the group as the composite unit) completed the project.

The concept of the “unit” becomes more crucial when solving proportional problems. For example, consider the following problem offered from Lamon (1993a):

On a recent business trip, 9 people traveled comfortably in two cars. Our company plans to send 18 sales representatives to a conference next week and I need to reserve some rental cars for their trip. How many rental cars should I reserve (p. 135)?

The first step to constructing a more sophisticated unit is to think of the nine people as one composite unit and the two cars as another composite unit. The level of complexity increases when the ratio, 9:2, is thought of as a new unit relating the number of cars to the number of people. Using 9:2 as the unit, we can form other, equivalent, multiple units. Doubling the unit, we get the relationship 18:4 = 2(9:2). The process of creating and utilizing complex units is one of the necessary tools for students to engage in proportional reasoning (Lamon, 1993a).
Additive versus Multiplicative Structures

Change is a concept addressed very early on in a student’s mathematical career (Behr, Harel, Post, & Lesh, 1992). For example, what change to the number 4 gives the number 8? Two ways in which the question can be solved are additively or multiplicatively. The additive structure gives that “adding 4” to the number 4 will result in the number 8. The multiplicative structure gives that “multiplying by 2” will transform 4 into 8. It is important to note that the additive relationship between 4 and 8 does not change if 9 is added to each of the numbers; specifically, in order to change from 13 to 17 we still “add 4”. In contrast, though, we cannot “multiply by 2” in order to change 13 into 17 (Behr, Harel, Post, & Lesh, 1992). In this situation, the additive relationship is invariant. It is crucial that students understand both additive and multiplicative structures and how to determine and utilize the appropriate relationship between two numbers to successfully reason proportionally.

Vergnaud (1988) stated that the “conceptual field of multiplicative structures consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide” (p. 141). There are numerous situations which lend themselves to multiplicative solutions. For example, simple proportions appear in problems involving ideas such as cost, sharing, and mixtures. Further, multiple proportions appear in problems involving such ideas as consumption and production of goods, space, mechanics, and heat. Each situation may have a different presentation from the previous and utilize a varied approach to find a solution. Therefore, students need experience with a rich and varied selection of multiplicative, proportional problems (Vergnaud, 1988).
**Qualitative Reasoning**

Thinking qualitatively about ratios and proportions is another important factor as students learn to reason about proportions quantitatively. Behr, Harel, Post, and Lesh (1992) suggested that information on students’ ability to think qualitatively illuminates their ability to think proportionally. Consider the equation $\frac{a}{b} = c$. A student who can think qualitatively about ratios should be able to answer questions such as: “what happens to $c$ if $a$ gets larger and $b$ stays the same?”, “what happens to $c$ if $b$ gets larger and $a$ stays the same?” etc. Similarly, an exploration of the questions such as “what happens to $c$ if both $a$ and $b$ increase?” or “what happens to $c$ if both $a$ and $b$ decrease?” illuminate a student’s qualitative thinking. Behr, Harel, Post, and Lesh (1992) suggested that the more comfortable a student is with reasoning qualitatively and recognizing when qualitative reasoning is not sufficient, the more successful the student will be when solving problems using proportional reasoning.

**Informal and Preproportional Strategies**

Students are sometimes able to solve proportional reasoning problems using informal strategies. These strategies often make sense and lead to correct answers, but prove insufficient as more complex problems are presented. Kaput and West (1994) stated there are three broad types of informal, or preproportional, solution strategies:

1. coordinated build-up/build-down processes
2. abbreviated build-up/build-down processes using multiplication and division
3. unit factor approaches (p. 244).
Coordinated build-up/build-down processes

This strategy is best illustrated through an example. Consider the placemat problem: A large restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used 35 pieces of silverware in its table settings last night, how many pieces of china did it use (Kaput, & West, 1994, p. 236)? A typical response using a build-up strategy is found in Table 2.1. The restaurant used 20 pieces of china last night.

Table 2.1. Build-Up Strategy

<table>
<thead>
<tr>
<th>Silverware</th>
<th>7</th>
<th>14</th>
<th>21</th>
<th>28</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>China</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
</tr>
</tbody>
</table>

Kaput and West (1994) described the set of steps generally taken when using a build-up strategy. These steps do not need to occur in this exact order. These steps are:

Initial conceptualization

1. Distinguish between the two referents A and B to be quantified in the problem solution.

2. Construct a semantic correspondence relation between the classes of referents A and B at a gross level.

3. Form units within each of the referents, A units, B units.
4. Construct either a correspondence relation between respective units at the group level (matching an A unit with a B unit), or construct a higher order group containing the A unit and the B unit as its two elements.

5. Distinguish between the third given quantity and the fourth, unknown, quantity by linking each to its respective referent type, A or B.

Computation

6. Increment (decrement) or skip count both quantities until the third given quantity is reached, coordinating on the basis of either the match between replicated A units and B units or on the basis of replications of the higher order unit consisting of the A unit and B unit joined together.

7. Identify its corresponding element of the other quantity as the problem’s solution (p. 247).

This approach uses additive strategies to solve a problem that is inherently multiplicative.

Abbreviated build-up process

The abbreviated build-up process is a slightly more advanced strategy than the original. The first five steps of the abbreviated process are identical to the original. Consider this slight variation to the placemat problem: A large restaurant sets tables by putting seven pieces of silverware and four pieces of china on each placemat. If it used 392 pieces of silverware in its table settings last night, how many pieces of china did it use (Kaput, & West, 1994, p. 236)? The initial conception of the problem is the same. The known information is that for every seven pieces of silverware that is used, four pieces of china are also used. I know that 392 pieces of silverware were used. Using the original strategy, another set of china and silverware would be consecutively added until
the total pieces of silverware were 392 and the solution would be the corresponding number of pieces of china.

The abbreviated solution: We have 392 total pieces of silverware with 7 pieces per placemat so, \( \frac{392\text{silverware}}{7\text{silverware/placemat}} = 56 \text{ placemats} \). We also know that there are 4 pieces of china per placemat so 56 placemats * 4 china/placemat = 224 pieces of china.

The abbreviated versions of steps 6 and 7 of the original are as follows:

6A. Divide the total given quantity by the quantity per unit to obtain the number of units.

7A. Multiply the number of units by the corresponding quantity per unit to determine the total unknown quantity (Kaput, & West, 1994, p. 249).

The major problem with using either the original or the abbreviated build-up strategy occurs when the unit increase is not a whole number. In the incrementing process, the unreduced “steps” may be too large to “hit” the target quantity. In the abbreviated version, the unit quotient may not be a whole number. Kaput and West (1994) stated the two most popular strategies to deal with this problem are:

1) To make a unit-size adjustment early in the process and then carry through with the process with the adjusted unit; and

2) To make an adjustment late in the process (p. 249).

The unit factor approach

The unit factor approach again utilizes the same initial five steps as both the build-up process and the abbreviated process. Consider this problem by Kaput and West (1994): To make Italian dressing you need four parts vinegar for nine parts oil. How much oil do you need for 828 ounces of vinegar (p. 246)? Again, the first five steps are
the same. We know that for every four parts of vinegar, we must use nine parts oil. We have 828 ounces of vinegar. Using the unit factor approach we would determine a single unit consisting of the amount of vinegar needed for one ounce of oil.

\[
\frac{4 \text{ vinegar}}{9 \text{ oil}} = \frac{4}{9} \text{ vinegar/oil}.
\]

Then, \( \frac{4}{9} \) vinegar/ounces of oil \(*\) 828 ounces of oil \( \approx \) 368 ounces of vinegar. The abbreviated versions of steps 6 and 7 are as follows:

6U. Divide the unit size of the unknown quantity by the unit size of the known quantity to determine the unit factor

7U. Multiply the unit factor and the given total quantity to determine the total amount of the unknown quantity (Kaput, & West, 1994, p. 251).

These three strategies lay the foundation for informal proportional thinking. Students must move beyond these proportional methods in order to solve more complex proportional problems.

Framework for Proportional Reasoning Semantic Types

There are many facets to successful proportional thinking. Categories, therefore, or groups of problems, must be created within this complex domain in order to study proportional reasoning. I will discuss two different categorizations.
Lesh, Post, and Behr (1988) arranged proportional reasoning tasks into seven different topic areas.

1. The first category contains the missing value problems. This set takes the form of

\[
\frac{A}{B} = \frac{C}{D}
\]

where three of the four values are given. The objective in this set of problems is to find the fourth missing value.

2. The second category contains the comparison problems. This set takes the form

\[
\frac{A}{B} \ (\geq ? \leq) \frac{C}{D}
\]

where all four values are given. The objective of these problems is to determine if the first ratio is greater than, less than, or equal to the second ratio.

3. The third category contains the transformation problems which take on one of two forms. The first transformation type problem addresses the direction of change judgments. The equivalence \( \frac{A}{B} = \frac{C}{D} \) is given and one of the four components is either increased or decreased. The objective is to determine whether the relationship (\(<, >, =\)) is true after the transformation. The second transformation type problem addresses the transformations to produce equality. In this type, the problem begins with this \( \frac{A}{B} < \frac{C}{D} \) relationship. Then, one of the variables, say A, must be transformed so that the new relationship \( \frac{A + x}{B} = \frac{C}{D} \) holds true.

4. The fourth category contains the mean value problems. In this set of problems the goal is to find the geometric or harmonic mean given two numbers. Given

numbers A and B the geometric mean is found using \( \frac{A}{x} = \frac{x}{B} \) and the harmonic
mean is found using \( \frac{A}{B} = \frac{A-x}{x-B} \). The harmonic mean is useful for finding the average rate of speed.

5. The fifth category contains the conversion problems. These problems involve conversions from ratios, to rates, to fractions. For example, given that the ratio of boys to girls in a class is 15 to 12, what fraction of the class are boys?

6. The sixth category contains problems with unit labels as well as numbers. In this set the student must not only be aware of the relationship between the number, but, also, between the units. For example, \( \frac{3 \text{ feet}}{2 \text{ seconds}} = x \text{ miles per hour} \).

7. The seventh and final category contains translation problems. In these problems, the student must take a given ratio and represent that same relationship using another representational system.

This arrangement of proportional reasoning problems is thorough, but too specific for my research interests. I am more interested in categorizing problems based on the situational relationships given within the context of a real-life situation. Lamon created a similar categorization scheme with fewer groups containing more problem types, and she created her categories based on the information given within a situational problem.

*Lamon*

Lamon (1993b) identified and described four proportional reasoning semantic type problems.

1. The first type is Well-Chunked Measures. These problems contain proportions with two different unit descriptions. For example, suppose a person is traveling for 3 hours and drives a distance of 150 miles. The problem gives two pieces of
information each bearing a different unit label. What is the average speed of the
car during the trip? To answer that question the student would need to create the
following ratio: \( \frac{150 \text{ miles}}{3 \text{ hours}} \). The student might then give the answer of 50 miles per
hour. In this situation the “miles per hour” is a well-chunked measure because the
combination of two common unit labels creates a third, distinct, common unit
label. The miles per hour unit is often referred to as speed. Another common well-
chunked measure is dollars per item which is commonly referred to as price.

2. The second type is Part-Part-Whole. These problems contain ratios of subsets
describing a whole. For example, one might describe a class as 12 boys to 10 girls
rather than using a description of 22 students. Another example might be
describing a set of test questions by how many were correct and how many were
incorrect. Problems comparing one subset to the whole also fall under this
semantic type. For example, one might describe a class as 12 boys out of 22
students.

3. The third type is Associated Sets. These problems contain proportions relating
two quantities that otherwise have no obvious relationship. The relationship
between the two quantities is defined within the context of the problem situation.
For example, a school might be buying pizza for a school dance. If one pizza
feeds three students then how many pizzas are needed to feed 57 students? The
relationship between pizza and students needed to be defined within the context of
the problem situation.

4. The fourth type is Stretchers and Shrinkers (which I will call Growth). These
problems contain a continuous ratio-preserving mapping between two quantities
representing a specific characteristic. For example, the specific characteristic could be a measure of distance such as height, length, width, or circumference. The problem situation could entail either scaling up or down, or stretching or shrinking. Suppose we have a rectangular picture on the wall that is 7 inches by 13 inches. If I increase the smaller side to 9 inches, and maintain the same relationship of length to width, what is the length of the longer side?

*Further exploration of Lamon’s guidelines*

I chose to further investigate Lamon’s (1993b) semantic structure. Using that structure as a framework, she conducted a study to determine:

1. Do the four semantic problem types result in a meaningful differentiation of student thinking about ratio and proportion?
2. What kinds of informal strategies do children use for these problems before instruction?
3. What instructional implications can be drawn from children’s preinstructional knowledge in relation to the four problem types?

The population in this study consisted of 69 male and 69 female sixth grade students from an urban middle school in Wisconsin and her population included a mix of low-income, middle-class, and upper middle-class families. No students received any formal ratio and proportion instruction before the study began. The study began with students answering an eight question written test consisting of two questions from each semantic type. The responses were broken into the following response categories:
1. at least one correct solution within each semantic type;

2. at least one correct solution for problems of each of the types: associated sets, part-part-whole, and well-chunked measures;

3. at least one correct solution for both part-part-whole and well-chunked measures;

4. at least one correct solution for a well-chunked measures problem;

5. no correct solutions; or

6. none of the above.

The preliminary results of the study found that 82% of the students fell in categories one through five. To further investigate the students’ thinking, two students were chosen, at random, from each of the six response categories for in-depth interviews.

During the interviews, each of which lasted approximately 2 hours, the students worked on 40 different problems representing each of the four semantic types. The students were able to use manipulatives, pictures, and charts. The problems were presented to the students in random order. The interviews were audio-taped and all written work was collected. The data were coded using the following mathematical dimensions:

a) use of relative or absolute thinking;

b) type of representation (verbal, pictorial, tabular);

c) quantity structure (singleton units, composite units); and

d) sophistication of the strategy (incorrect strategy, preproportional reasoning, qualitative proportional reasoning, or quantitative proportional reasoning).
The preproportional reasoning label was assigned when a student used informal methods which resulted in correct solutions of scalar and functional relationships. Once the solution strategies were coded, similarities were investigated within each semantic type. If eight of the ten semantic type problems fit specific strategy predictors, the entire type was assigned a label as shown in Table 2.2. Table 2.3 gives the number of each level of response within the four different semantic types.

Table 2.2. Sixth-Grade Students’ Strategies for Solving Ratio and Proportion Problems

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Nonconstructive strategies</strong></td>
<td></td>
</tr>
<tr>
<td>Avoiding</td>
<td>No serious interaction with the problem</td>
</tr>
<tr>
<td>Visual or additive</td>
<td>Trial and error or Responses without reasons or Purely visual judgments (“It looks like…”) or Incorrect additive approaches</td>
</tr>
<tr>
<td>Pattern building</td>
<td>Use of oral or written patterns without understanding numerical relationships</td>
</tr>
<tr>
<td><strong>Constructive strategies</strong></td>
<td></td>
</tr>
<tr>
<td>Preproportional reasoning</td>
<td>Intuitive, sense-making activities (pictures, charts, modeling, manipulating) and Use of some relative thinking</td>
</tr>
<tr>
<td>Qualitative proportional reasoning</td>
<td>Use of ratio as a unit Use of relative thinking and Understanding of some numerical relationships</td>
</tr>
<tr>
<td>Quantitative proportional reasoning</td>
<td>Use of algebraic symbols to represent proportions with full understanding of functional and scalar relationships</td>
</tr>
</tbody>
</table>
Table 2.3. Number of Sixth Graders using each Strategy by Semantic Type

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Well-Chunked Measures</th>
<th>Part-part-whole</th>
<th>Associated Sets</th>
<th>Growth</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avoiding</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>Visual or additive</td>
<td>5</td>
<td>3</td>
<td>0</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>Pattern building</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Preproportional reasoning</td>
<td>11</td>
<td>19</td>
<td>0</td>
<td>1</td>
<td>31</td>
</tr>
<tr>
<td>Qualitative proportional reasoning</td>
<td>1</td>
<td>1</td>
<td>23</td>
<td>3</td>
<td>28</td>
</tr>
<tr>
<td>Quantitative proportional reasoning</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(Lamon, 1993b, p. 47)

Lamon found that the different semantic type problems elicited different levels of proportional reasoning. She noted that achievement in one semantic type did not appear to influence achievement in another. She further concluded that students most often used ratio and proportions to solve associated sets problems. Students, in general, did not use proportional reasoning strategies to solve part-part-whole type problems, but, rather, used additive building up strategies. Students were not able to identify the multiplicative nature of the growth problems (Lamon, 1993b).

This study was limited to sixth grade students with no formal ratio and proportion instruction. The population in my study is decidedly different, high school aged students who have all completed Algebra I with a grade of at least a “C”. The framework of Lamon’s study was useful in the design of my study. The four semantic type problems
encompass many aspects of proportional reasoning and allow for different problem solving strategies.

*Langrall and Swafford*

Langrall and Swafford (2000) used the semantic framework structure described in the previous study working with sixteen middle school students. They investigated Lamon’s findings that:

1. Students tend to use a higher level of proportional reasoning strategies in solving problems of associated sets.
2. Students use more informal solution strategies when solving part-part-whole relationships even when they have demonstrated higher level strategies on other problems.
3. Many students do not see the proportional relationships involved in the well-chunked measures semantic type (for example, that “miles per hour” is the number of miles traveled every hour).
4. Growth problems are the most difficult for students possibly because they are the least conducive to pictorial representation (p. 255-6).

In this study, sixteen students were selected, four from each of the grades five through eight. The students were chosen by their teachers and one from each grade level had a low mathematical understanding, two had an average level, and one student had a high level of mathematical understanding. The students participated in interviews completing problems from each of the four semantic types identified by Lamon (1993b).
Using the literature and the results of the student interviews, Langrall and Swafford (2000) categorized the student responses into one of four levels:

1. **Level 0: Nonproportional reasoning**
   - Guesses or uses visual clues (“It looks like…”)
   - Is unable to recognize multiplicative relationships
   - Randomly uses numbers, operations, or strategies
   - Is unable to link the two measures

2. **Level 1: Informal reasoning about proportional situations**
   - Uses pictures, models, or manipulatives to make sense of situations
   - Makes qualitative comparisons

3. **Level 2: Quantitative reasoning**
   - Unitizes or uses composite units
   - Finds and uses unit rate
   - Identifies or uses scalar factor or table
   - Uses equivalent fractions
   - Builds up both measures

4. **Level 3: Formal proportional reasoning**
   - Sets up proportion using variables and solves using cross-product rule or equivalent fractions
   - Fully understands the invariant and covariant relationships.

The students participating in the study received varying levels of ratio and proportion instruction. The researchers did not report the number of students solving problems at each level, although they described at least two student responses from each level.

**Summary**

Proportional reasoning has been shown to take many forms. Successful proportional thinking requires unitizing and relative thinking. Research has shown that students initially use informal strategies, such as simultaneous build-up of measures, to solve proportional problems. Since proportional reasoning is such a large mathematical
domain, researchers have determined ways to categorize the many different problem
types. Chapter 3 describes the way I have utilized previous research to create a design to
answer my research questions.
CHAPTER 3
RESEARCH METHOD

Introduction

In this chapter I discuss the method used in this study. First, I reiterate the research questions. The second section contains a description of the Promising Youth program, the participants in the study and their schools. Third, I describe the research methods used for data collection. A discussion of the data analysis procedures concludes this chapter.

Research Questions

The population of students participating in this study comes from the poorest counties in a particular southeastern state where many students do not complete high school. My research seeks to answer the following questions:

1. Do students from rural, low-income school environments who have completed Algebra I (and Geometry) have advanced proportional reasoning skills?

2. Can students from rural, low-income school environments who demonstrate an ability to solve proportions algorithmically solve problems that do not fit into the cross multiply and divide mold (do they understand, conceptually, the mathematics behind the algorithm)?

3. Can high school students from rural, low-income school environments model proportional problems using concrete materials? If so, how does that modeling influence their thinking?
These questions are investigated using a systematic exploration of the proportional reasoning abilities of the participants in the study using observations and interviews.

Participants

Promising Youth

The Promising Youth Partnership Program began in 2002 at a large public university in the southeastern part of the United States. For the purposes of this study we refer to this school as Southeastern University. One goal of the Promising Youth program is to encourage students to attend college who might not otherwise. This program does not target the highest achieving students, but those who have potential and need encouragement. High school students apply in their freshman year and begin the program that summer. Each summer, for three consecutive years, the Promising Youth participants spend time at Southeastern University. The rising sophomores live at Southeastern for one week, the rising juniors for two weeks and the rising seniors for three weeks. During their stay the students take classes, eat at the dining halls, and experience college life. For many, it is the first time they have left home or seen any college. Ideally, after the students complete the three years in the program, they are better academically prepared to succeed in post secondary education. Also, the students gain confidence and determination needed to apply and attend college.

The Promising Youth students participate in many different classroom situations during their time at Southeastern, although this study focuses solely on the mathematics portion of the program. During the students’ first summer they participate in classes involving writing, reading, mathematics, SAT preparation, navigating the library, using computers, and a team-building, outdoor ropes course. During the second summer, the
students participate in classes involving reading, writing, mathematics, introduction to science, and art. In their third and final summer, the Promising Youth complete a three-week long college simulation. The students enroll in five courses (mathematics, English, lab science, social science, and art) run like college classes. The students complete assignments, take tests, and receive grades for each course.

During the school year, while the students are attending their own high schools, they maintain a connection to Southeastern University. The program director visits each school at least once during each semester. Further, several members of the Promising Youth math staff visit each high school at least once a year. In addition to the visits, all Promising Youth students receive math challenge problems to complete approximately every five to six weeks during their school year. The problems are sent to their homes, completed by the students, and mailed back to Southeastern University. The evaluation of these problems consists of comments and suggestions (grades are not given), picking the top three responses, and mailing the results back to the students. The top students in each problem set are recognized.

Forty-five students began the program in 2002. Thirty-two of those forty-five students attended the program for all three of the summers finishing in 2004. Those students completed their senior years in high school in the 2004-2005 academic year. Over half of that class applied to Southeastern University. Five other students applied to other higher educational institutions. One student joined the Air Force and two joined the National Guard. Overall, eighty-eight percent of those students had post-secondary plans. The college attendance is particularly impressive considering these students come from
high schools where the dropout rate is very high (between 14% and 50%) and post-secondary education is rare.

Schools

The initial school selection and participant selection originated with the program director. Five of the six schools participating in Promising Youth were selected because they were located in the poorest counties in the state. Sterns, the sixth high school, was selected because of previous ties to the University and because of a diverse population. A portion of the school is middle to upper-middle class, but another part occupies a low socioeconomic status.

Once the schools were selected, the program director contacted the school counselors to discuss the purpose of the program. The program is not designed for the top achieving students, but for those students with college potential, who need additional academic or psychological support to make the final commitment. The program seeks to motivate students to attend college who might not otherwise. Depending on the school, the school counselors, teachers, or a combination nominate ninth grade students for the program. The students send an application, including several essays, and a grade report, to the program director at Southeastern University. Promising Youth accepts approximately fifty new rising sophomores each summer. The number of applicants largely exceeds the number of available slots. In Summer 2005, there were over 110 applications for fifty positions.

Five of the six target high schools participating in the Promising Youth program are located in Hanover, Barton, and Applewood counties in this southeastern state. These three counties rank among the lowest in per capita income in the state. Specifically,
Applewood County has the lowest per capita income in this state. The sixth school, Sterns High School, resides in a relatively affluent county, but has a large population (about one-third) of students on free/reduced lunch program.

The statistical information, summarized in the following three paragraphs, for all of the schools, except Sterns High School, was obtained from each school’s District Profile (www.myscschools.com) and reflects data collected during the 2003-2004 school year. The information for Sterns High School was obtained from its 2004 report card (www.myscschools.com). Sterns High School, although maintaining a diverse population, is located within an affluent county and, therefore, the statistics in that district profile were misleading as a representation of Sterns. Hence, I chose to use the report card only for Sterns High School and not the entire county.

In Hanover County, Wilks High School and Eastover High School were selected. Wilks, in the Hanover 1 District, has approximately 63% of its students receiving subsidized meals, an average annual teacher salary of $38,759, an average total (math plus verbal) SAT score of 985, and a dropout rate of 22%. The population of students is approximately 56% African American and 43% white. Out of graduating students from this district, approximately 63% attend a two or four year college and 33% take employment or other options. Eastover High School, in Hanover 2 District, has over 87% of its students receiving subsidized meals, an average annual teacher salary of $38,756, an average total (math plus verbal) SAT score of 809, and a dropout rate of 32%. The population of students is approximately 95% African American and 5% White. Out of graduating students from this district, approximately 60% attend a two or four year college and 40% take employment or other options.
In Barton County, Bishop and Danville were selected. Bishop, in Barton 1 District, has approximately 65% of its students receiving subsidized meals, an average annual teacher salary of $39,668, an average total SAT score of 955, and a dropout rate of 17%. The population of students is approximately 60% African American, and 40% white. Out of graduating students from this district, approximately 62% attend a two or four year college and 38% take employment or other options. Danville, in Barton 2 District, has approximately 93% of its students receiving subsidized meals, an average annual teacher salary of $36,972, an average total SAT score of 852, and a dropout rate of 50%. The population of students is approximately 98% African American, 2% White. Out of graduating students from this district, approximately 55% attend a two or four year college and 43% take employment or other options.

The following is the information for the final two schools. The district containing Applewood, Applewood District, has approximately 90% of its students receiving subsidized meals, an average annual teacher salary of $37,606, an average total SAT score of 798, and a dropout rate of 17%. The population of students is approximately 95% African American, 4% White, and 1% Hispanic. Out of graduating students from this district, approximately 56% attend a two or four year college and the remainder join the job market or pursue other options. Sterns High School has approximately 30% of students on subsidized meals, an average annual teacher salary of $40,576, an average total SAT score of 1001, and a dropout rate of 37%. The population of students is approximately 60% African American, 20% Hispanic, and 20% White.
According to the guidelines found on the National Food and Nutrition website (http://www.fns.usda.gov/cnd/governance/notifications/IEGs03-04.pdf), to qualify for free lunch a family of four must have an annual salary of no more than $23,920 and to qualify for reduced price lunch a family of four must have an annual salary of no more than $34,040. In the entire state, approximately 51% of students are receiving subsidized meals, the average annual teacher salary is $40,362, the average SAT score is 989, and the dropout rate is approximately 13%. Of graduating students from high school, 63% enter a two or four year college and approximately 32% begin employment or other options (www.myscschools.com). The preceding statistical information is summarized in Table 3.1.
# Table 3.1. Population Statistical Information

<table>
<thead>
<tr>
<th>School</th>
<th>Subsidized Meals (Percentage)</th>
<th>Ave Teacher Salary (Dollars)</th>
<th>Ave. Total SAT (math plus verbal)</th>
<th>Dropout Rate (%)</th>
<th>Population Breakdown</th>
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</thead>
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<tr>
<td>Wilks HS</td>
<td>63</td>
<td>38,759</td>
<td>985</td>
<td>22</td>
<td>56 - African Am 43 - White</td>
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<td>Eastover HS</td>
<td>87</td>
<td>38,756</td>
<td>809</td>
<td>32</td>
<td>95 – African Am 5 – White</td>
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<tr>
<td>Sterns HS</td>
<td>30</td>
<td>40,576</td>
<td>1001</td>
<td>37</td>
<td>60 – African Am 20 - Hispanic 20 – White</td>
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<td>Danville HS</td>
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<tr>
<td>Applewood HS</td>
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<td>37,606</td>
<td>798</td>
<td>17</td>
<td>95 – African Am 4 - White 1 – Hispanic</td>
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<tr>
<td>State Average</td>
<td>51</td>
<td>40,362</td>
<td>989</td>
<td>13</td>
<td>41 – African Am 54 – White 4 - Other</td>
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</tbody>
</table>
Data Collection

The data collection occurred during June and July of 2005. The group of rising seniors spent three weeks, the rising juniors spent two weeks, and the rising sophomores spent one week at Southeastern University.

Between two and four days were spent during the program concentrating on problems from the different semantic types outlined by Lamon (1993b) in “Ratio and Proportion: Connecting Content and Children’s Thinking:” part-part-whole, associated sets, well-chunked measures, and growth (stretchers and shrinkers). On a typical day, the students worked in small groups on one proportional reasoning activity. A more complete description is provided below.

Observations

On at least two separate days, as a class, we spent 30-45 minutes of the Promising Youth 90-minute class period working on proportional reasoning problems. There was one rather involved, inquiry-type problem given. The class was broken into groups of three or four and the students worked on problems that were created to illicit discussion with many appropriate solution strategies. Some of the small groups were videotaped during the duration of the activity. During most of these sessions, I was conducting interviews, so other math teachers worked with the groups. All teachers working with the groups used similar prompts to make the experience as uniform as possible for the students.

The following, adapted from www.figurethis.org/challenges/c52/challenge.htm, was given for the day focused on the Part-Part-Whole semantic type. The class began with a discussion of a technique used to estimate the population of bass fish in a
particular lake. Catching and counting all of the bass fish in the lake would be difficult and require an enormous amount of time and effort. Therefore, estimation techniques are useful. This estimation procedure begins by catching a certain number (say 100) of fish, marking them, and releasing them back into the lake. Several days later, after the fish have returned to their normal swimming patterns, a sample of fish is caught. The investigators count the number of tagged fish and the number of total fish caught. That information is used to estimate the total number of bass fish in the lake.

In the class activity, navy beans sitting in a big bowl were used to represent the fish in a lake. Several beans, representing the tagged fish, were marked by the students using Sharpie markers. The following is the activity given to the students:

Capture-recapture is a statistical method used to estimate the size of a population. Fish and wildlife management experts, demographers, and scientists use this and other techniques to find the number of people or animals in a region.

1. Discuss the tagging technique.
2. Locate the classroom population of “bean fish.”
3. Each group will “capture” approximately 20 – 30 bean fish and tag them (you can use the markers provided).
   a. My team tagged this many bean fish:
4. Each group will place the “tagged” bean fish back into the large population.
   a. The total number of bean fish tagged by the class:
5. In your group devise a plan to use the “tagged” fish to estimate the entire population of bean fish. Describe the procedure below.
6. Share the plans as a class and decide on the most effective plan.
7. Each group takes three samples of the large population. Estimate the size of the entire bean fish population using the class plan.
   a. Sample 1
      Population estimate:
   b. Sample 2
      Population estimate:
   c. Sample 3
      Population estimate:
8. Average the three population estimates from your group.
   Average group estimate:
9. Combine the results from each individual group to create a class population estimate.

Class estimate:
10. How do your estimates compare with the other groups? How do your estimates compare with the class estimate? Why might these numbers be different?
11. Suppose you wanted to estimate the population of deer in your county. Describe the procedure you would use (without capturing all of the deer).
12. Why does your procedure work?

The class was split into groups of three or four and worked on the problem. The groups were arbitrarily chosen based on the students’ seats. The teachers’ initial questions were:

- Do you understand the problem?
- What is the question asking?

Each group was responsible for marking their bean fish. The total number of marked fish in the bowl was determined. The students, in their groups, completed the remainder of the activity. During the process, the students were video-taped and their final written answers were collected. The process was similar for all four of the class proportional reasoning activities. The remaining three activities are found in Appendix A.

**Interviews**

The majority of data was gathered through the interview process. The interviews were conducted during the same 90-minute class periods. I worked with individual students on the same set of eight proportional reasoning problems (two from each semantic type). Each interview lasted between 45 and 70 minutes. Students were encouraged to think aloud, and I tried to illicit as much explanation from the students as possible. I retained all of the students’ work and these interviews were also videotaped.

The purpose of the interviews was two-fold. First, I wanted the students to showcase their individual talents rather than their group participation. Further, I believed that I was able to more completely understand the students’ thought processes through individual
conversations related to specific mathematical proportional reasoning problems. An example of one question used during the interview along with sample prompts is provided below.

The following is an activity, utilized during the interview portion of the research, based on a problem by Lamon (1993b). The original problem was used in a research experiment with sixth grade students. The question falls into the Part-Part-Whole construct.

In Chester, the demand for apartments was analyzed, and it was determined that to meet the needs of the community, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose the builder is planning to build a large apartment complex containing between 35 and 45 apartments. Exactly how many apartments should he or she build to meet this regulation? How many one-bedroom, two-bedroom, and three-bedroom apartments will there be?

The students were presented with the problem and had the option of using connect blocks to represent the apartments. They could also use paper and pencil for pictorially representing the situation or to complete calculations. The questions and prompts were the following:

- Please describe what you are thinking/doing.
- What does the situation look like?
- How can you use what you know to answer the question?
Students were encouraged to persist until they completed the problem or became too frustrated to continue. The remaining interview questions are found in Appendix B.

The interviewees were selected before the Promising Youth summer session began. The first step in selecting the interviewees from the rising seniors and the rising juniors was to use the students’ pre-assessment test taken during their first summer at Southeastern University. I chose three students, at random, from each of the upper and lower quartiles of scores and four students from the middle two quartiles. Ten students were selected in that manner from the rising seniors and nine students from the rising juniors. The rising sophomores did not have pre-assessment test scores because it was their first year in the program. They were only at Southeastern for one week and needed to take pre-assessment tests during one of the three mathematics meetings. Due to time constraints, I randomly selected only two students for interviews.

Data Analysis

Organization

The purpose of the study is to categorize the population’s proportional reasoning abilities into levels. Utilizing Langrall and Swafford’s (2000) level designation, I separated the problems first into the four distinct proportional reasoning categories:

1. part-part-whole
2. associated sets
3. well-chunked measures
4. growth.

I evaluated the responses to each problem separately as to the level of proportional thinking demonstrated (between levels zero and three).
The proportional reasoning was evaluated based on the following scale taken from Langrall and Swafford (2000) outlined in Table 3.2.

Table 3.2. Langrall and Swafford’s Proportional Reasoning Scale

<table>
<thead>
<tr>
<th>Level</th>
<th>Strategies Exhibited</th>
</tr>
</thead>
</table>
| **Zero:** Nonproportional Reasoning | a) Guesses using visual clues  
                  b) Unable to recognize multiplicative relationships  
                  c) Random usage of numbers, operation or strategies  
                  d) Unable to link the two measures  
                  e) Does not lead to correct solutions or development of more mature proportional reasoning |
| **One:** Informal Reasoning about Proportional Situations | a) Uses pictures, models, or manipulatives to make sense of situations  
                  b) Makes qualitative comparisons |
| **Two:** Quantitative Reasoning | a) Unitizes or uses composite units  
                  b) Finds and uses unit rate  
                  c) Identifies or uses scalar factor or table  
                  d) Uses equivalent fractions and builds up both measures |
| **Three:** Formal Proportional Reasoning | a) Sets up proportion using variables and solves using cross-product rule or equivalent fractions  
                  b) Explains completely the invariant and covariant relationships |

I watched the videos at least two times of the students working on the interview questions and made a tally mark under “occurrences” each time I witnessed evidence of a particular strategy. Table 3.3 exhibits the rubric used for the first segment of analysis.
Table 3.3. Proportional Reasoning Evaluation Rubric

<table>
<thead>
<tr>
<th>Level</th>
<th>Strategies Exhibited</th>
<th>Occurrences</th>
</tr>
</thead>
</table>
| **Zero:**  
 Nonproportional Reasoning | a) Guesses using visual clues |  |
|  | b) Unable to recognize multiplicative relationships |  |
|  | c) Random usage of numbers, operation or strategies |  |
|  | d) Unable to link the two measures |  |
|  | e) Does not lead to correct solutions or development of more mature proportional reasoning |  |
| **One:**  
 Informal Reasoning about Proportional Situations | a) Uses pictures, models, or manipulatives to make sense of situations |  |
|  | b) Makes qualitative comparisons |  |
| **Two:**  
 Quantitative Reasoning | a) Unitizes or uses composite units |  |
|  | b) Finds and uses unit rate |  |
|  | c) Identifies or uses scalar factor or table |  |
|  | d) Uses equivalent fractions builds up both measures |  |
| **Three:**  
 Formal Proportional Reasoning | a) Sets up proportion using variables and solves using cross-product rule or equivalent fractions |  |
|  | b) Fully understands the invariant and covariant relationships |  |
I transcribed and coded segments of their interviews using a coding scheme from Lamon (1993b). This coding was:

- Use of relative or absolute thinking
- Type of representation (verbal, pictorial, tabular)
- Quantity structure (singleton units, composite units)
- Sophistication of the strategy (incorrect strategy, preproportional reasoning, qualitative reasoning, quantitative reasoning).

After coding the interviews, their written work was reviewed and each question was given a level designation. For levels zero through two, if a student exhibited one of the strategies indicated for the level, I tentatively assigned them to that level, pending further investigation of the written work. To designate a student to level three, I needed evidence of their ability both:

1. to set up a proportion using variables and solve using cross-product rule or equivalent fractions; and
2. to fully understand the invariant and covariant relationships.

The rationale behind that decision was to avoid allowing students to set up a proportion and use the “cross multiply and divide” algorithm without understanding the mathematical principles behind the process.

Some students began the problems using a level one strategy, such as drawing pictures. If they continued with the problem and exhibited level three behaviors, such as translating those pictures into equivalent fractions and solving the problem formally, they were categorized as a level three. Five of the interviews were analyzed in the same manner by a second mathematics educator to check for reliability.
Summary

The following describes how the data analysis was used to answer the research questions.

- **Question 1:** Do students from rural, low-income school environments who have completed Algebra I (and sometimes Geometry) have advanced proportional reasoning skills?

For the purposes of this study, I defined levels two and three as “advanced” proportional reasoning. If students exhibited level two or three strategies in at least three of the four construct types, they were categorized as advanced. This was done for each of the types of proportional reasoning discussed earlier.

- **Question 2:** Can students from rural, low-income school environments who demonstrate an ability to solve proportions algorithmically solve problems that do not fit into the cross multiply and divide mold (do they understand, conceptually, the mathematics behind the algorithm)?

This answer to this question was found in the interviews. If students defaulted to the algorithmic “cross multiply and divide,” I probed to determine if the students understood the mathematics behind the process. Further, many problems selected for the interviews did not lend themselves to the standard procedure. Therefore, students' reasoning abilities were challenged and exposed.

- **Question 3:** Can high school students from rural, low-income school environments model proportional problems using concrete materials? If so, how does that modeling influence their thinking?
During the interview process, concrete manipulatives were available for the students to use. If students struggled, I encouraged them to model the situation using the manipulatives. The results are found in Chapter Four.
CHAPTER 4
RESULTS

Introduction

In this chapter, I examine the data collected during this study and discuss the results. The chapter begins with a brief discussion of the descriptive statistics in my sample and a depiction of the research setting. I detail the techniques used, as discussed in Chapter 3, to analyze my data. Each research question is individually addressed in the final section.

Descriptive Information

Statistics

I collected data for the study during the summer of 2005 at a large university in the Southeastern United States, referred to as Southeastern University. During that summer, 107 students were enrolled and participated in a program, referred to as Promising Youth, designed to allow and encourage students from rural, low-income secondary school environments to attend college.

Of those 107 students, 21 took part in the in-depth interview portion of this study. Two of the interview participants were rising sophomores, nine were rising juniors, and ten were rising seniors (henceforth referred to as sophomores, juniors, and seniors, respectively). Fifteen of the participants were female and six were male. The original intent was to have an equal number of male and female participants, but the interview was voluntary for the students. Many of the targeted male interviewees in the group of seniors were unwilling to participate in a one-on-one interview. Therefore, the percentage
of female participants in the interview group was larger than the percentage of females in
the entire population. Out of the twenty-one interviewees, seventeen were African-
American, one was Hispanic, and three were Caucasian. The participants were from each
of the six schools targeted by the Promising Youth program; one from Applewood High
School, two from Bishop High School, four from Danville High School, four from
Eastover High School, two from Sterns High School, and eight from Wilks High School.

To begin selection of the interviewees, each class was divided into quartiles based
on their previous summer’s mathematics pre-test score. The pre-tests were given to all of
the Promising Youth participants on the first day in the math class. The test evaluates
basic mathematics competency. A copy of this test can be found in Appendix C.

For the seniors, I randomly selected three students from both the upper and lower
quartiles and four students from the middle 50%. For the juniors, I randomly selected
three students from each of the upper and lower quartiles and three students from the
middle 50%. Since I did not have any prior information on the sophomores, I randomly
chose two students from the entire population.

I pre-selected all of the individuals to participate in the interviews, but was unable
to get complete participation. Seven of the students in the senior class opted not to
participate in the process because the interviews took place during the designated math
instruction time; and the participants would miss class during the interview period. Two
students indicated that class was too important to miss, others felt uncomfortable, and
others were not interested in, participating in a one-on-one interview.

An effort was made to maintain diversity in my sample. My sample for the
seniors consisted of one student from the lower quartile, five students from the middle
50%, and four students from the upper quartile. My sample for the juniors consisted of two students from the lower quartile, four students from the middle 50%, and three students from the upper quartile. Overall, my sample is skewed slightly towards the higher-performing students (as measured by the previous year’s pre-test score). This leaning may yield somewhat more positive results than a completely representative sample.

*Setting*

The interviews took place in similar environments. I was able to interview the seniors and sophomores in the same room at Southeastern University. I used a room in the same building, but on a different floor, to conduct the interviews with the juniors. I utilized a long table for each of the interviews and I sat next to the participant on one side of the table. All of the manipulatives and other necessary instruments were located on the table (paper, scissors, connect blocks, liquid measuring cups, pitcher of water, cut-outs of the pizza question, calculator, and pencils). All sessions were videotaped and the interviews were performed with only two people in the room, the interviewee and me. I interviewed each of the participants one time for 45 – 70 minutes. Due to the time restrictions of the Promising Youth program, the interviews ended when the allotted math time was completed for the day. Some subjects were not able to attempt all eight of the interview questions during that time period. If an interviewee did not attempt one of the questions, I used N/A for the score. This situation is discussed in more detail later in this chapter.
Evaluation

I transferred all of the video information from mini-DV tapes onto DVDs. Each interview became an individual DVD. Each interview was viewed at least twice and evidence of the interviewee’s level of proportional reasoning abilities was recorded on the worksheet created (see Chapter 3). I also logged responses that were typical of the group or responses that were exceptional, relative to the rest of the interviewees. I noted when a participant used manipulatives while working on an interview question and used the student’s written work to help determine the specific level of proportional reasoning exhibited for each interview question.

As discussed in Chapter 3, each interview participant received a total of eight interview questions (exceptions noted). There were two questions from each proportional reasoning construct: Growth, Well-Chunked Measures, Part-Part-Whole, and Associated Sets. Each response was assigned a score from zero (indicating non-proportional reasoning) to a three (indicating formal proportional reasoning). An independent mathematics educator viewed five of the interview DVDs and assigned level designations to each question based on the same rubric used by the researcher. The scores on each of the questions matched my scores. The grading scheme, as discussed in Chapter 3, was taken from Langrall and Swafford’s (2000) level designation as found in Three Balloons for Two Dollars. I compiled the responses in Table 4.1. Each number corresponds to the student’s assigned proportional reasoning level on that question.
Table 4.1. Individual Question Results

<table>
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<tr>
<th>Student</th>
<th>Grade</th>
<th>Sex</th>
<th>Growth 2 - A</th>
<th>WCM 2 - B</th>
<th>Assoc Sets 2 - C</th>
<th>PPW 2 - D</th>
<th>Growth 1 - E</th>
<th>PPW 1 - F</th>
<th>WCM 1 - G</th>
<th>Assoc Sets 2 - H</th>
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| mean    | 0.90  | 1.52 | 1.33 | 1.52 | 1.33 | 1.63 | 0.78 | 1.36 |
| median  | 0     | 2    | 1    | 2    | 1    | 2    |      |      |

Analysis of Data

Introduction

To answer my three research questions, I analyzed the data gathered from the in-depth interviews. I analyzed each of the eight interview questions separately. The student responses to each individual interview question were analyzed independently and assigned a level between zero and three, inclusively. Each of the 21 interviewees was given eight different interview questions (exceptions noted above). Levels two and three are defined as “advanced.” The following discussion examines the results obtained from each of the interview questions.
These two rectangles are the same shape.

Find the height of the larger rectangle.

Figure 4.1 Interview Question A

All twenty-one participants attempted this question. Fourteen responses were assigned a level zero, no responses were assigned a level one, two responses were assigned a level two, and five responses were assigned a level three. The mean assigned level was 0.905 and the median assigned level was 0.
Table 4.2 Question A Summary

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<tr>
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</table>

The responses to this particular question fell into one of two categories: those who recognized the multiplicative relationship between the two rectangles and those who did not. For many students, their initial response was to impose an additive relationship to the situation. Since 12 feet was four more than 8 feet, the missing side, according to their reasoning, had to be four more than 6 feet. Therefore, the most typical incorrect response to this question, given by 13 participants, was 10 feet.

Another error made by one student on this problem was to hold the areas of the two rectangles constant. This method did not lead to a correct solution of the problem. The following is an excerpt from the interview with Jose, a senior from Eastover High School, the student who attempted to solve the problem by holding the areas of the rectangles constant. His response was categorized as a level zero response.

Researcher (R): OK. So what did you do?

Jose (J): [After concluding that the answer was 36 feet] Basically what I did was multiplied this in front of there to get this one (indicating the first rectangle).

R: OK.

J: Then I put it in the formula, I forgot what’s the name of it, but I put it in there [the formula for the area of a rectangle]. I got 8 feet times 6 feet equals twelve feet times x.
So, basically what I did was add, well multiplied, the 8 feet and 6 feet and got 48 feet squared equals 12 feet plus, I mean times, \( x \). And so I just subtracted 12 from both sides and it gave me 36 feet equal to \( x \).

R: Why did you subtract there?

J: Because, um, really um, really I need the \( x \) value to be by itself on one side.

R: OK. OK and so what did you keep the same about these rectangles?

J: What I kept the same is feet, the measurements.

R: OK and by setting 8 times 6 equal to 12 times \( x \), what were you keeping the same?

J: The same numbers. Those numbers that they gave me. That is how to find \( x \).

Jose incorrectly determined that in order to solve for “\( x \),” he should set the areas equal. During subsequent questioning, I determined that the only formula Jose was familiar with was the area formula and that was the reason for the choice.

The following is an excerpt from an interview with Amanda, a junior from Danville High School. This excerpt is an exemplar of a typical response from a student answering using level zero strategies. Thirteen participants answered in a similar manner.

Researcher (R): OK. How did you get 10 feet?

Amanda (A): [After concluding that the answer was 10 feet] I added 4 to 6 because 4 plus 8 is 12.

R: Does that keep the same, um; does that keep the relationship between the sides the same?

A: Does it keep the relationship the same?

R: Uh-huh.

A: Yes.
Amanda incorrectly recognized the relationship between the sides of the rectangles as an additive rather than as a proportional, or multiplicative, relationship. This was the most common error of all interviewees. The response was rated as a level zero because she was unable to recognize the multiplicative relationship, and her thoughts did not lead to correct solutions or development of more mature proportional reasoning.

Five of the participants successfully recognized the proportional relationship between sides of the rectangles. The following is an excerpt from an interview with Jacqueline, a junior from Danville High School, who exhibited sound proportional reasoning on this problem. This excerpt is a typical response from a student answering using level three strategies.

Researcher (R): All right. Tell me what you did.
Jacqueline (J): [After concluding that the answer was 9 feet] Well, I did a proportion. I put 8 over 12 and 6 over \( x \). And that \( x \) is 9 because 8 goes into 72 nine times.
R: And how did you know to put it in a proportion like that?
J: [laughing] Because, ummmm, that’s the only thing that I thought about doing.
R: How did you know they were related using a proportion?
J: Because they’re similar.
R: OK.

Jacqueline exhibited some knowledge that the ratio between two corresponding sides of similar shapes remains constant. She did not think additively about this problem. This answer was rated as a level three because she set up a proportion using variables and understood the relationship between the variables.
Interview Question B: Well-Chunked Measures

You get the following “special” offer in the mail for a monthly magazine subscription:

A. a 6-month subscription for 2 payments of $6.00 each – OR -
B. a 9-month subscription for 3 payments of $6.00 each – OR -

1. Is option A a better deal than option B? Why or why not?

2. The magazine also offers a 12-month subscription for 4 payments of $5.00 each. Is it a better deal to subscribe for 12 months? Why or why not?

Figure 4.2. Interview Question B

All 21 participants attempted this question. Two responses were assigned a level zero, seven responses were assigned a level one, eleven responses were assigned a level two, and one response was assigned a level three. The mean assigned level was 1.524 and the median assigned level was 2.

Table 4.3. Question B Summary

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<td>1</td>
<td>4.8%</td>
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</table>
Many interviewees had difficulty comparing the three different payment options since the subscriptions were for different lengths of time. Only 5 out of the twenty-one successfully calculated a unit-rate without any prompting.

The following is an excerpt from an interview with Aisha, a senior from Bishop High School. This excerpt is a typical response from a student answering using level one strategies. She is able to make qualitative comparisons between the options, but cannot determine, mathematically, which is the best choice.

Researcher (R): Can you just tell me what you, how you got your first answer there?
Aisha (A): I picked option A because, ummm, because I think the payments is (sic) cheaper than the nine month subscription.
R: OK. So how much are you paying for the six months?
A: Ummm, you’re paying 12 all together.
R: OK. And how about for the nine month?
A: Eighteen.
R: OK. Now, you think the six months for 12 dollars is cheaper than the nine months for 18 dollars?
A: Yes. So you should pick the cheaper one.
R: OK. [Aisha moves to the second part of the question]
A: For the second one I put yes because you get a year’s subscription of the, um, magazine and it’s better than option A and you only, um, and you only get 6 months’ subscription.
R: OK, but how about, how do the prices compare?
A: The twelve month is 20 dollars all together, but this one [pointing to option A] is cheaper cause it’s 12 dollars.

R: Do you get more magazines in one of the options?

A: This one [indicating the 12 month option].

R: OK. This one costs more, but you get more magazines. And this one costs less, but you get less magazines. Is there a way to tell which one is cheaper for the amount of magazines that you get?

A: I don’t think so.

R: OK. So which one do you think is the best out of all these three?

A: The twelve month…Because the 12 month you get a year’s worth and with option A you only get six month’s worth. Even though you pay more for this one, but I think you are getting worth your money.

Aisha felt the most comfortable thinking absolutely. One subscription was “cheaper” so that was the one she wanted. The relationship between price and number of magazines was not important in her decision. This response is a level one because she makes qualitative comparisons.

The following is an excerpt from an interview with Isaac, a junior from Danville High School. This excerpt is an exemplar of a typical response from a student answering using level two strategies. Eleven students, or 52.4% of the students, answered exhibiting level two reasoning strategies.

Researcher (R): OK. So what did you think for that one?

Isaac (I): I think they are the same. They cost the same.

R: How did you get that?
I: Because it says 6, it costs two payments of $6 each for six months for the first one. And three payments of $6 for the nine month subscription.

R: OK

I: And it’s about the same because you paying $6 for three and $6 for two.

R: OK. So how did you figure out they were the same?

I: Oh, because it’s $12 for six months and $18 for nine months and 6 times 2 is 12 and 9 times 2 is 18.

R: So what does that 2 mean?

I: It’s a payment.

R: Well for six months it costs $12. That’s what you told me. And it costs $18 for nine months. You got that by multiplying 9 times 2. Why did you multiply 9 times 2 to get 18 dollars?

I: That’s how much it costs per month.

R: So they cost the same per month?

I: Yeah. [Moves onto the second part]

R: OK. So what do you think for the next one?

I: Oh I think the 12 month subscription is better than the six month and the nine month because it says for 12 months you pay $5, 4 payments of $5, and that’s $20.

R: OK.

I: And you pay a dollar and seventy cent per month and for options A and B you pay 2 dollars per month.

R: So the twelve month then is cheaper?

I: Yes.
Isaac’s reasoning serves as an example of the reasoning exhibited by most level-two interviewees. Those participants found a “unit rate,” either cost per month or cost per three months. Since these students utilized a unit rate, they exhibited level two reasoning strategies.

Only Calia, a sophomore from Applewood High School, exhibited level three strategies to solve this problem. She utilized and understood a formal proportion to solve the problem. Understanding and using that formal proportional algorithm is what differentiates a level two response from a level three response. In my research, both level two and level three responses are considered “advanced” because they recognize and utilize the proportional nature of the problem. Her written work is found in Figure 4.3.

Part of the text of the interview follows:

Calia (C): [Discussing the second part of the question] What I did was, basically, said 12 months, well cross multiplying makes things easier for me when it has to deal with money, so I cross multiplied. Twelve months equals 20 dollars basically because 4 payments of 5 dollars each, so 20 dollars. So 12 over 20 was the fraction I came up with and for this side. I put the 1 beside the 12 because I wanted to find out how much I’m paying each month. And I put an \(x\) beside the 20 because I don’t know how much money I pay each month. So, \(12x=20\) which means that \(x\) equals 20 over 12. I divided and got 1.6, but 6 was going to keep going, so I just rounded it off to 1.67.
In the first part of this question, Calia used proportions to determine that six dollars for twelve months was the same ratio as nine dollars for eighteen months. In the second part of this question, Calia employed proportions again to compute the “per month” cost for each of the offers. She correctly determined that one would pay approximately $1.67 per month on the 12-month plan and $2.00 per month on either the 12- or 18-month plans. She comfortably used a formal proportion, understood why a proportion worked, and cross multiplied to solve for the correct solution exemplifying level three reasoning abilities. Her solution was classified as a level three because she used a formal method for solving the problem (a proportion) and understood why that formal method led to the correct solution.
Interview Question C: Associated Sets

Would a girl or a boy get more pizza?

The Girls

The Boys

Figure 4.4. Interview Question C

All 21 participants attempted this question. Two responses were assigned a level zero, eleven responses were assigned a level one, seven responses were assigned a level two, and one response was assigned a level three. The mean assigned level was 1.333 and the median assigned level was 1.

Table 4.4. Question C Summary

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No participants were immediately able to distinguish if a boy or a girl would have more pizza. The most common strategy was to break the pizzas up into slices and assign each person a particular number of slices. At first, ten of the interviewees wanted to ignore the leftover pieces or save them for later.

The following is an excerpt from an interview with Tunisia, a junior from Wilks High School. This excerpt is an exemplar of a typical response from a student answering using level one strategies. She is able to make qualitative comparisons, but cannot support her conclusions with mathematics.

Tunisia (T): I’m just trying to see if, do I have to work this out, you know?

Researcher (R): You can work it out if you want. You can do it any way that you want. We can just look at the pictures and try to figure it out that way.

T: The boys?

R: You think the boys? How come?

T: Because it’s less, boys having to share, um, one pizza. Where there’s more girls having to share two pizzas. So it wouldn’t be, like say, for this one here [pointing to one of the girl’s pizzas] one pizza is not filling all of the girls and two pizzas, you can’t go around a second time. There’s not going to be more. There’s going to be more for them [indicating the boys] and less for them [indicating the girls].

R: OK. Um, just talk a little bit more about what you mean by “they won’t be able to go around a second time.”

T: The boys, say there’s just like three of them, they’ll most likely take up about less than half of the pizza. They might be able to go back for seconds, but the girls…
R: Let’s say they took the most they could on their first trip. How much would the boys get to take?

T: I’d say about two.

R: Two pieces?

T: At least.

R: How about the girls? Let’s say they took all they could on the first trip.

T: They might be able to, they might, they might take up one pizza and then they would share the second one.

R: Well if there are three boys sharing one pizza, then each boy could have as much as one-third of the pizza. How about the girls?

T: I guess it would be about two-sevenths. I don’t know.

R: They WOULD get two-sevenths. Can you compare two-sevenths and one-third?

T: Two-sevenths. It goes in there at least two times [dividing two into seven and checking on her calculator].

T: 3.5? Yeah 3.5.

R: So what does 3.5 mean?

T: Three and a half. So each girl gets three and a half. The boys each got two. The girls get more.

Tunisia’s response is typical of a student answering the question using level one strategies. She attempted to break the pizzas up into pieces and distribute those pieces. Most students broke the pizza up into eight slices and could not evenly distribute those eight slices among the boys and girls, leaving them incapable of answering the question.
She is unable (even with guidance from the researcher) to utilize the one-third versus two-sevenths relationship. She can only make qualitative comparisons using pictures.

The following is an excerpt from an interview with Trevor, a senior from Wilks High School. This excerpt is an exemplar of a typical response from a student answering using level two strategies.

Trevor (T): The boys get more.

Researcher (R): OK. Why do the boys get more?

T: They got one pizza and there’s three of them and if you put three girls with this one [indicating one of the girl pizzas] and three girls with this one [indicating the other girls’ pizza] then you still have one girl left and ….

R: She’s got to eat.

T: Yeah. So she takes some away from each pizza.

Trevor was able to create composite units. Three girls were a unit. He then compared each of his composite units to the number of pizzas and found that the boys would have more pizza. All of the students who successfully completed this problem using level two strategies answered similarly.
If is $\frac{2}{3}$, draw the whole.

If is $\frac{2}{3}$, draw $\frac{1}{3}$

If is $\frac{2}{3}$, draw $\frac{1}{3}$

If is $\frac{2}{3}$, draw 1

Figure 4.5. Interview Question D

All 21 participants attempted this question. Four responses were assigned a level zero, four responses were assigned a level one, eleven responses were assigned a level two, and two responses were assigned a level three. The mean assigned level was 1.524 and the median assigned level was 2.
Table 4.5. Question D Summary

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Most interviewees did not understand the question upon first reading it. I used an example story to illustrate the nature of the question. The story used was similar to the following: Suppose you were moving and went to the store to buy boxes. The boxes were big and heavy so on the way to the car you dropped some of them. When you got to the car you found that you still had four boxes. Those four boxes are only two-thirds of the number of boxes you bought in the store. Can you figure out how many boxes you bought? Most students understood the question after some explanation.

The following is an excerpt from an interview with Jose, a senior from Eastover High School. This excerpt is an exemplar of a typical response from a student answering using level two strategies. Using manipulatives, he was able to answer the question correctly.

Jose (J): So what do you want us to draw?

Researcher (R): I want you to draw the whole.

R: [After Jose drew six boxes] How did you get that?

J: Well, you have four here, and that’s two-thirds what you had already bought, and you need one-third, one-third to make it a whole. So, really what I did was, if this was two-thirds, if you, hold up. If you cut it off right here [drawing a line dividing the four squares into two sets of two] then that would be half of what you bought, right?
R: I’m not sure how you got that. Just explain that to me.

J: OK.

R: Now you’ll also notice that a lot of times I’m not going to say right or wrong, so you may be right, you may be wrong. I just want you to explain.

J: Say for instance, you have this [two boxes] and you put this inside your car or your house. Then you would only have this [two boxes] so you would have one-half.

R: Right. Well, that would be one-half of this [indicating the four boxes]. Right?

J: Yeah.

R: Is one-half of these four the same thing as one-half of the whole?

J: No. No, not really. But the way I got it is, like, I don’t really know how to explain it. It’s really like you got two-thirds of this, ok, and you need one more third to actually make it to a whole. So, what I really did was add one-third more to get the whole.

R: And how did you figure out what was one-third?

J: How I figured it out was, well if I, I kind of look at it like, I don’t know. Like a pizza. [Draws two-thirds of a pizza]…I don’t really know how to explain.

R: Let’s try the next one. Maybe you’ll be able to explain the next one.

[After more confusion we get out the connect blocks]

R: Let’s say these [four] blocks represent two-thirds. How many “thirds” are in “two-thirds”?

J: Two.

R: OK. So, I’ve got 2 one-thirds here. So how much is “one-third”?

J: Two boxes.

R: So how many more boxes should we take to get a “whole”?
J: [Takes two more boxes] Two more.
R: Does that make sense?
J: Uh-huh.
R: So what’s my whole?
J: Those six boxes. [Successfully continues the problem using the manipulatives]

Jose was unable to comprehend two-thirds until he had a manipulative (and a little instruction). Most students who initially had difficulty with this problem were able to successfully complete it after using the connect blocks. Most students, like Jose, used two blocks as their “unit.” They understood that three “units” made a whole.

The following is an excerpt from an interview with Jana, a sophomore from Bishop High School. This response highlights one student who was able to complete the problem without the use of manipulatives. She was one of only three level two students who correctly answered the question without the aid of manipulatives or additional instruction. Additionally, there were two students, using level three strategies, who did not require the use of manipulatives or additional instruction.

Researcher (R): Can you figure out how many boxes I bought in the store?
Jana (J): Twelve.
R: How did you get 12?
J: It’s not 12. If all four of these boxes are two-thirds then you had six.
R: How did you get six?
J: Because if you only had two boxes, well two boxes is one-third. If two boxes is one-third and you have these boxes right here is two-thirds then we have another third then it’s two boxes. So you have six boxes.
I am making cookies and my recipe calls for 2 cups of milk, 1 tablespoon of butter, and $\frac{1}{2}$ cup of sugar. Unfortunately, I only have $1 \frac{1}{2}$ cups of milk. I need to use less butter and sugar so that the cookies still taste good. How much butter and sugar should I use?

Figure 4.6. Interview Question E

All 21 participants attempted this question. Four responses were assigned a level zero, eight responses were assigned a level one, seven responses were assigned a level two, and two responses were assigned a level three. The average assigned level was 1.333 and the median assigned level was 1.

Table 4.6. Question E Summary

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As with the first Growth question (Question A), fifteen of the participants initially viewed this situation additively. They noted that the milk decreased by $\frac{1}{2}$. Most students’ first instinct was to subtract $\frac{1}{2}$ from the other two ingredients. When that strategy left no sugar for the cookies, some took a multiplicative approach. For those who then viewed the situation as a percentage decrease in milk, they ran into another stumbling block when “taking away” 25% of $\frac{1}{2}$ cup of sugar.
The following is an excerpt from an interview with Latoya, a senior from Wilks High School. This excerpt is an exemplar of a typical response from a student answering using level one strategies. She was able to determine that she needed “less,” but was unable to correctly identify how much “less” she needed.

Researcher (R): So, what should I do with the other two ingredients?

Latoya (L): Divide them in half.

R: OK. So why should I divide them in half?

L: Cause you have one and one-half of the milk and you have less of that.

R: You definitely have less. Do I have half of what I need?

L: You have a little bit more than half.

R: A little bit more than half, yeah. Can you figure out exactly how much?

L: More than half.

R: Can you figure out how much sugar then?

L: I say one-third.

R: And how much of the tablespoon? I don’t want to use the whole tablespoon.

L: Half of that.

She is able to recognize that she needs to take “less” of the other two ingredients. She has no numerical basis for deciding how much “less” she needs. The interview continues with manipulatives to model the “whole” amount of milk and the “reduced” amount of milk. She does, however, recognize that this is not an additive relationship. She does not want to take the same absolute numerical value away from each ingredient. Other students who answered using level one strategies had similar responses.
The following excerpt is an excerpt from an interview with Lela, a senior from Wilks High School. This is a typical response from a student answering using level two strategies.

Researcher (R): OK. Tell me what you did here.

Lela (L): Um, I said that the milk went down. It was, I first put a half, because they took a half from it, then I realized that it was only a fourth of the whole entire cup. And then it said that everything else has the same measurements. So I took a fourth from each of the other items and I ended up with three-fourths of the butter and one-fourth of the sugar.

R: OK.

L: Cause it was a half a cup of sugar and you have to take one-fourth of that. That would be one-fourth.

R: A fourth of a half is one-fourth?

L: [long pause] Half. It’s half. [Struggled to find one-fourth of one-half]

Lela was able to recognize that the milk was reduced by one-fourth (25%). She then applied that scale factor reduction to the other ingredients. She, like all of the students using level two strategies, still struggled to find one-fourth of the half cup of sugar.
Interview Question F: Part-Part-Whole

In Chester, the demand for apartments was analyzed and it was determined that to meet the needs of the community, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose the builder is planning to build a large apartment complex containing between 35 and 45 apartments. Exactly how many apartments should he or she build to meet this regulation? How many one-bedroom, two-bedroom, and three-bedroom apartments will there be?

Figure 4.7 Interview Question F

Nineteen out of the 21 participants attempted this question. Three responses were assigned a level zero, two responses were assigned a level one, thirteen responses were assigned a level two, and one response was assigned a level three. The mean assigned level was 1.632 and the median assigned level was 2.

Table 4.7. Question F Summary

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Many of the interviewees were able to answer the question correctly using a variety of build-up strategies. Some literally drew boxes to represent the apartments, some added the numbers together until they reached a sum between 35 and 45, and some made the first set of apartments (8) the unit and determined how many “units” were
needed. Three students confused the number of apartments with the number of bedrooms. I spent a few minutes with each of these interviewees clarifying the question.

The following is an excerpt from an interview with Kristin, a junior from Wilks High School. This excerpt is an exemplar of a response from a student answering using level two strategies.

Kristin (K): I don’t know.

Researcher (R): Tell me what you did so far.

K: I just wrote down the information that I was given.

R: OK.

K: And I divided by that [indicating the number 8] to see.

R: All right. How did you get that eight?

K: I added four plus three plus one.

R: Right. And so what does that eight mean?

K: That’s, um, all the apartments that are going to be on one floor?

R: Maybe like in a floor or a building or something. OK, so you’ll have eight apartments every time you build. I need between 35 and 45 apartments now.

K: It’s not going to equal out.

R: What do you mean by “equal out”? You mean it doesn’t divide equally?

K: Uh-huh.

R: I see you divided 8 into 35. OK, so you can’t get exactly 35 apartments, building it this way. But I don’t need exactly 35 apartments. I need somewhere between 35 and 45.

K: OK. Uh, five apartment buildings, right?

R: How did you get that?
Charlie was driving on a highway containing a speed trap. The speed trap works in the following way: a camera records the time each car drives past a particular point on the highway. Another camera records the time each car passes a second point 10 miles down the road. The average speed of the car over that 10 mile stretch is then automatically calculated. The driver of a car is issued a speeding ticket if his average speed is above the 55 mph speed limit.

Just when Charlie entered the speed trap he noticed a police car on the side of the road. The police officer pulled out behind Charlie. Not wanting to get a ticket, Charlie immediately slowed down and drove a constant 40 miles per hour. After five miles the police officer gave up and stopped following Charlie. Since the police officer turned around, Charlie began driving faster. Charlie passed the second checkpoint on the highway and was pulled over. The computer calculated that his average speed was 53 miles per hour for the entire 10 mile stretch of road. Surprisingly, the police officer who pulled Charlie over gave him a ticket for driving 79 miles per hour.

Charlie went to court. How did the officer prove that Charlie was guilty?

Kristin made a composite unit out of the three different apartment types. Her unit became a set of eight apartments. All students utilizing level two strategies answered similarly. Most students then “built-up” their composite units until they found a number between 35 and 45. Many interviewees also needed guidance to get to the number between 35 and 45, as Kristin did.

*Interview Question G: Well-Chunked Measures*

Nine out of the 21 participants attempted this question. The remaining 12 participants did not have enough time to try this question because they spent more time on previous questions. Four responses were assigned a level zero, three responses were assigned a level one, two responses were assigned a level two, and no responses were assigned a level three. According to the research, I arranged the questions in ascending order of difficulty (Lamon, 1993b). Based on their performance on prior questions, I
believe that those who were unable to complete this question, due to a lack of time, would have scored either at a level zero or one. I believe none of the participants who did not get to this question would have achieved an “advanced” (level two or three) on this question. For those students who did attempt the problem, the mean assigned level was 0.778 and the median assigned level was 1.

Table 4.8. Question G Summary

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There was only one interviewee who was able to begin working on this problem without aid. The students were unable to make use of the important pieces of information in the question. Interestingly, during the process of working with the participants on Question G, I discovered that many have little or no idea how speed is measured. I transcribe one of these conversations below.

The following is an excerpt from an interview with Rolanda, a senior from Eastover High School. In this example, I try to ascertain the student’s level of understanding of miles per hour, used commonly to measure speed. This student was not able to make sense of the problem and answered using only level zero strategies.

Researcher (R): What do we know about what’s going on? What does the problem tell us? What do we know about what Charlie’s doing?
Rolanda (Ro): That he was speeding.

R: For the whole time?

Ro: Until he seen the police man and then he slow down.

R: OK. And then what did he do?

Ro: Slow down.

R: And how fast was he driving?

Ro: 40 miles an hour.

R: And how long did he do that?

Ro: Uh, for five miles.

R: And then what happened?

Ro: He started speeding again.

R: OK. And how do you know that he started speeding again?

Ro: I don’t know.

R: What do the cameras tell us?

Ro: Um, that his speed was 53 mph for the entire 10 mile stretch.

R: So, what did he do for the first five miles?

Ro: Drove 40.

R: OK. Well if his average speed over the whole thing…How do you find speed, average speed?

Ro: Goodness…

R: Well, if you don’t remember an equation, that’s ok. Let’s just think about it.

Ro: Would it be the time over the rate, the rate over the time, or rate times time, or…it’s something.
R: This is what happens when you’re just trying to remember a formula. Let’s see. What are my units for speed?

Ro: Ummm.

R: When I say he was going 40, forty what?

Ro: Miles per hour.

R: OK. What does miles per hour mean?

Ro: The speed. How fast you going.

R: OK. And so, if I was driving 40 miles per hour, um, how far would I go in two hours?

Ro: I don’t even know that.

R: How about if I told you that I drove 80 miles in 2 hours. How fast was I going, on average?

Ro: I don’t know.

R: All right. Let me write this down for you. [If I drove 25 miles in 1 hour, how fast was I going, on average?]

Ro: Oh my goodness. I don’t know. I forgot how to do this.

R: Well let’s talk about miles per hour. What does that mean?

Ro: What do you mean, what does it mean?

R: Like, if I’m driving 60 miles per hour, what does that mean? I know you look down at your speedometer and it says “60”, but what does it really mean? What does it tell me?

Ro: How fast you going.

R: How far would I go in an hour, if I was driving 60 miles per hour?

Ro: I don’t know.
Rolanda appeared to become flustered throughout our discussion of this question which suggests she might have known slightly more than she was able to communicate. Throughout my experience dealing with Rolanda, however, during the interview and whole-class activities, I believe that this conversation gives an accurate representation of her knowledge. On many occasions, she behaved similarly, and I believe that the “flustered” reaction is a reflection of her personality and the interview accurately depicts her understanding of speed.

Five of the interviewees showed a similar lack of knowledge regarding “speed.” Rolanda was unable to make any progress toward a solution because she was lacking necessary understanding of the well-chunked measure commonly called “speed.”

The following is an excerpt from an interview with Alana, a senior from Wilks High School. This excerpt is a typical response from a student using level one strategies. She showed a stronger understanding of “miles per hour,” but was not able to utilize that knowledge to answer the question.

Alana (A): He was going 40 miles per hour for the first five miles.
Researcher (R): And how about for the second five miles?
A: He was going approximately 79 miles per hour.
R: Well that’s what the officer is saying. He was going 79 miles per hour for the second five miles.
A: Right.
R: But there was no camera to catch him. There was no cop to catch him going that fast. So, how can he show that he was doing the 79 miles per hour? What do we know about his total average speed?
A: It’s only 53 miles per hour.

R: If we know his average speed over the ten miles is 53 miles per hour and he went 40 mph for the first five miles, what can you tell me about his speed for the second five miles?

A: Ummm.

R: Was he going faster or slower than 53?

A: He had to be going faster.

R: Is there a way to figure out how much faster?

A: Ummm…

R: Well they found average speed. How do you find average speed? Or speed? Let’s say your speedometer is broken, but want to know how fast you are driving. Tell me another way to find out how fast you are going.

A: Wouldn’t you time yourself for like a mile?

R: OK. And then, what would that tell you?

A: Like for about one minute or so, you could…

R: Tell me what miles per hour means.

A: Like how many miles you go in an hour.

R: OK. So tell me a way to find out my miles per hour if my speedometer is broken.

A: You could calculate your time, um, how much time it takes to do one mile times the minutes?

Alana has a vague idea about speed, or average speed, but is not able to apply that knowledge to the problem. She understands that Charlie had to be traveling faster than 53
miles per hour for the second five miles. She, like most of the interviewees, is not able to find average speed given the information in the problem.

The following interview showcases a student who fully understands “miles per hour” and successfully utilizes that information to solve the problem. Roland, a senior from Wilks High School, exhibits level two proportional reasoning strategies when solving this problem.

Roland (Rl): One argument would be that…it says here that the speed limit is 55 and if the average speed is above 55 it’s a ticket. So, it’s a ten mile stretch. So he starts driving 40 miles per hour, then he speeds up. OK. I think the argument here would be that although his average speed came out to be 53 miles per hour for the ten mile stretch, he still could have been speeding when he got to the second point. So, maybe that’s how he was ticketed. Even though they look at the average between these two points, the officer might have been looking at the speed of the last point.

Researcher (R): OK. What makes you think he is going faster at the last point?

Rl: Well, for one, it says that “Charlie began driving faster.” So, if you look at that and you look at his average speed, in order for the average to be around 53, there has to be higher speeds and lower speeds because here you see that it was 40 miles per hour and he would have had to go faster than 53 in order for it to average back, you know, to 53, with the 40 miles per hour.

R: How did the officer know it was 79? Let’s say that he doesn’t have a radar gun. All they had was the time that he started, the time he was done. That’s how they got the average speed over the whole thing. How would he have known it was 79 and maybe not 100?
Rl: I think they also looked at the time it took.
R: OK and what does that have to do with it?
Rl: Across this 10 mile stretch, I guess it’s supposed to take you so long, you know, in
order to get through that stretch.
R: How would you find the average speed over a 10 mile stretch?
Rl: You’d go by the distance and then the time it took to get through that.
R: And what do you do with those two things to find speed?
Rl: You divide, if I’m not mistaken. OK. So you go, um, over the ten mile stretch, um.
R: Do we know how long it took him to go those 10 miles?
Rl: No, it doesn’t say.
R: Do we know what his average speed was?
Rl: That was 53.
R: So, is there a way we can figure out how long it took him to go over those 10 miles?
Rl: You can work backwards. So, to find the average speed you take distance divided by
time and so that would give you the average speed. So, here you have the average speed
so you have to divide that by the distance of ten miles.
R: Let me just check this with you. You have the average speed which is measured in
what kind of units?
Rl: Miles per hour.
R: And what are the units for distance?
Rl: Miles. [works for a while] I don’t think this is right. It’s giving me a decimal of an
hour.
R: Well, just think about it. You’re driving 10 miles. How much time does it usually take you, or your friends, or people you know to drive 10 miles?

RL: Well, the decimal means that it’s a part of an hour. It takes me about 15 or 20 minutes to drive 10 miles. So that’s not a whole hour.

R: Would you feel better if the time was in minutes.

RL: Yeah, so I’ll multiply by 60. So, from what I’ve calculated here it’s about 11 minutes.

R: OK. So what do we know then?

RL: OK. For this ten mile stretch his average speed was 53 miles per hour and it took him about 11 minutes to get there.

R: So, can you tell me how long it took him to drive any particular part of those 10 miles?

RL: OK. For the first 5 miles he drove 40 miles per hour. [Does some calculations] So, it took him seven and a half minutes to do the first five miles. So, since the whole thing took him about 11.3 minutes, and for the first part it was 7.5, that means he took 3.8 minutes, roughly, or roughly 4 minutes, to do the other five miles.

R: Can we prove he really should get a ticket?

RL: OK. So the last stretch was five miles and it took him roughly, well this [the calculator] says 3.8, so I’ll round that to four minutes. We want to find the average speed. So, five miles in four minutes. Let’s see. [Divides 5 by 4]. That doesn’t seem right.

R: Why don’t you go back and check what you did.

RL: Oh. OK. I think I see what I did. I substituted that for hours [the four minutes], but that is really minutes. So, I have miles per minute. Um. [To himself] How much is a half an hour? So, it’s about a sixth of an hour? No. It’s not quite a sixth. [Does some more
calculations and divides four by sixty]. That’s what it is! That’s what it is! Four minutes of an hour would equal to that [points to 0.0666666 on his calculator].

R: What’s the unit on that number?

RI: That would be hours. So 5 divided by that [0.066666] I got about 75.

R: So, with your estimation, what do you think?

RI: I could give Charlie a ticket for 75. And so that’s how you can prove it. By his average speed, I mean by his average time during the last five miles, is how he got the ticket for 79 miles per hour.

Roland was able to use the distances and speeds given in order to find the necessary unit of time. He was then able to use time to find speed. Also, he successfully converted between minutes and hours. He understood the relationships between the distance, time, and speed, and, with a little help, was able to solve this difficult problem. Only one other interviewee was able to use level two strategies and solve the problem.
Two candles are made of different kinds of wax. They are of equal length and they are lighted at the same time, but one candle takes 9 hours to burn out and the other takes 6 hours to burn out. After how much time will the slower-burning candle be twice as long as the faster-burning candle?

Eleven of the 21 participants attempted this question. Two responses were assigned a level zero, three responses were assigned a level one, six responses were assigned a level two, and no responses were assigned a level three. Again, based on the performance of the participants on prior questions, I believe that those who did not have enough time to attempt this question would have scored either at a level zero or one. For those who did attempt the problem, the average assigned level was 1.364 and the median assigned level was 2.

Table 4.9. Question H Summary

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None of the students was able to begin to solve the problem the way it was worded which implied that they were unable to think algebraically. Therefore, I suggested assigning a specific length to the candles (18 inches) as a strategy to solve the problem.
Five of the interviewees were able to calculate the amount of the candle burning away each hour. One student was able to correctly solve the entire problem independently using the connect blocks.

The following is an excerpt from an interview with Amanda, a junior from Danville High School. She was unable to solve the problem and exhibited level zero strategies during the process.

Amanda (A): I don’t know.

Researcher (R): OK. Tell me what you’re reading in the problem.

A: That there are two candles. They’re equal length and one takes nine hours to burn and the other takes six. And they’re asking me “How much time till the slower burning candle is twice as long as the faster burning candle?”

R: Well, first let’s talk about which candle is the slower burning and which candle is the faster burning.

A: The nine hour is slower.

R: Right. What if I told you that each candle started out at 18 inches? So, we’ve got two candles and they are both eighteen inches. One of the candles takes nine hours to completely burn out. The other one takes six hours to completely burn out. Does that help to answer the question?

A: [After thinking for a long time] Five?

R: After five hours?

A: Uh-huh.

R: And what made you say five?

A: I divided nine into 18 and got two. I divided six into 18 and got three.
R: And what does that tell you? The two and the three – what do those numbers mean?
A: [shrugging] Just something I guessed. I added those numbers together and got five.

Amanda made no progress towards solving this problem. She calculated the amount of candle burning each hour, but did not recognize that number. She made a wild guess, after doing random calculations, at a solution. Amanda did not exhibit any strategies that would move her towards a solution. Her response is categorized as a level zero.

The following is an excerpt from an interview with Brenda, a senior from Wilks High School. This excerpt is a typical response from a student answering using level two strategies.

Researcher (R): Let’s say each candle was 18 inches. Can you figure out how much would burn each hour?
Brenda (B): OK. Eighteen inches and six hours. Eighteen times six hours? No. We want to find the rate. So, it would be three inches an hour.

R: How much would be burned in one hour?
B: OK. In one hour out of the eighteen inches there would be fifteen inches left.

R: Good. Now how did you get that?
B: Just subtract three from eighteen.
The process continued in this manner for Brenda until she determined when the slower-burning candle was twice as long as the faster-burning candle. All interviewees who used level two strategies were able to determine a unit rate for both candles. There were varying degrees of success utilizing those rates to answer the question.
Research Questions

Question 1

My first research question is “Do students from rural, low-income school environments who have completed Algebra I (and Geometry) have advanced proportional reasoning skills?” Overall the interviewees did not exhibit advanced proportional reasoning skills. I defined “advanced” to be exhibiting level two or three strategies, for both questions, in at least three of the four construct types. Table 4.10 summarizes the data of each question. Table 4.11 summarizes the data according to construction type. Table 4.12 summarizes the data according to the percentage of responses for each construction type. Table 4.13 summarizes the data, comparing non advanced and advanced scores, according to construction type. Table 4.14 summarizes the data, comparing non advanced and advanced scores, according to the percentage of responses according to construction type.

Table 4.10. Number of Responses per Question at each Level

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Table 4.11. Number of Responses per Question Arranged by Construction Type

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Table 4.12. Percentage of Responses per Question Arranged by Construction Type

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<td>9.5</td>
<td>52.4</td>
<td>33.3</td>
<td>4.8</td>
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<td>H</td>
<td>18.2</td>
<td>27.3</td>
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<tr>
<td>Well-Chunked Measures</td>
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<td>33.3</td>
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</tr>
<tr>
<td></td>
<td>E</td>
<td>19.1</td>
<td>38.1</td>
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<td>9.5</td>
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</table>
Table 4.13. Number of Advanced versus Nonadvanced Responses by Construction Type

<table>
<thead>
<tr>
<th>Construction Type</th>
<th>Question</th>
<th>Number of Level 0 and Level 1 Responses: Non Advanced</th>
<th>Number of Level 2 and Level 3 Responses: Advanced</th>
<th>Total Number of Responses</th>
</tr>
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<tbody>
<tr>
<td>Part-Part-Whole</td>
<td>D</td>
<td>8</td>
<td>13</td>
<td>21</td>
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<td></td>
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<td>21</td>
</tr>
<tr>
<td></td>
<td>H</td>
<td>5</td>
<td>6</td>
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<tr>
<td>Well-Chunked Measures</td>
<td>B</td>
<td>9</td>
<td>12</td>
<td>21</td>
</tr>
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<td></td>
<td>E</td>
<td>12</td>
<td>9</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 4.14. Percentage of Advanced versus Nonadvanced Responses by Construction Type

<table>
<thead>
<tr>
<th>Construction Type</th>
<th>Question</th>
<th>Percentage of Level 0 and Level 1 Responses: Non Advanced</th>
<th>Percentage of Level 2 and Level 3 Responses: Advanced</th>
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<tr>
<td>Part-Part-Whole</td>
<td>D</td>
<td>38.1</td>
<td>61.9</td>
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<td></td>
<td>F</td>
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<td>66.7</td>
<td>33.3</td>
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<tr>
<td></td>
<td>E</td>
<td>57.1</td>
<td>42.9</td>
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</table>

The participants exhibited stronger proportional reasoning abilities in the part-part-whole construct. They are weakest in the growth (stretchers and shrinkers) construct. Table 4.15 summarizes each participant’s data.
Table 4.15. Question Response Data for Each Participant

<table>
<thead>
<tr>
<th>Student</th>
<th>Part-Part-Whole</th>
<th>Advanced in P-P-W</th>
<th>Associated Sets</th>
<th>Advanced in AS</th>
<th>Well-Chunked Measures</th>
<th>Advanced in WCM</th>
<th>Growth</th>
<th>Advanced in Growth</th>
<th>Advanced Overall</th>
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</tbody>
</table>
The data show that two of the 21 interviewees meet the criteria for demonstrating “advanced” proportional reasoning abilities. Many students showed advanced abilities in one of two questions in a particular construct. For example, 57% of the participants used advanced proportional reasoning skills to answer the first question in the Well-Chunked Measures construct successfully, but only 22% were able to solve the second question successfully (and many did not have time to complete the question) using advanced proportional reasoning skills. Therefore, only two out of the twenty-one participants (or 9.5%) qualified as advanced in the Well-Chunked Measures construct.

**Question 2**

The second research question is “Can students from rural, low-income school environments who demonstrate an ability to solve proportions algorithmically solve problems that do not fit into the cross multiply and divide mold (i.e., Do they understand, conceptually, the mathematics behind the algorithm)?” To answer this question, I examined interview data for examples of students using the cross multiply and divide mold. Surprisingly, only two out of the 21 interviewees correctly used the algorithm to solve a problem. Those are the same two students who qualified for the “advanced” proportional reasoning label from Question 1 of this dissertation. Calia’s work is found in Figure 4.10. The following dialogue is an excerpt from an interview with Calia, a sophomore from Applewood High School.
Researcher (R): OK. So, how did you get nine?

Calia (C): Cross multiplication.

R: Tell me how you set up this [indicating the proportion] right here.

C: Ok. Since 6 was the, I’m going to say that was the height…I have the height for this one and it’s 6. I made that the numerator and I don’t have the numerator for this one so I put x. And for my denominator I put the length and that’s 8 feet. I put the 8 with the 6 because it’s for that [indicating the top rectangle] rectangle and not this one. Now, 12 is
for the other rectangle so I put that below the $x$. When I cross multiplied I got $8x=72$ and since it’s an equation I divided by 8 and got 72 over 8 and divided that and got 9.

Calia clearly understood how to utilize a proportion to answer the question and was not just “cross multiplying and dividing.”

The data collected during the interviews pertaining to the question “Can students from rural, low-income school environments who demonstrate an ability to solve proportions algorithmically solve problems that do not fit into the cross multiply and divide mold?” is somewhat misleading. The two students who correctly used and understood proportions exhibited a strong ability to reason proportionally in every question. The mere fact that they utilized an algorithm to expedite the solution shows their deep understanding of the proportional nature of the questions. The vast majority of the interviewees did not recognize that these questions involved proportional relationships. Therefore, they did not set up a proportion and just “cross multiply and divide.” Hence, those students who performed the algorithm understood why it worked and were not just going though a set of rote steps (although it is hard to make a valid conclusion since the data for this question are limited to two students).

There is also an alternate explanation for the pattern in my data. In developing this study an assumption was made that the students would have basic proportional reasoning skills and recognize the multiplicative nature of the interview questions. This assumption proved false in that, as discussed previously, many students were unable to recognize the underlying relationships in these problems. All students who participated in this study have been taught the “cross multiply and divide algorithm,” although only two were able to utilize it in this situation. The data suggest that teaching algorithms without
the corresponding conceptual understanding does a disservice to the student. Algorithms taught in such a manner may be memorized, but are not understood. Further implications are discussed in Chapter 5.

**Question 3**

My third research question is “Can high school students from rural, low-income school environments model proportional problems using concrete materials? If so, how does that modeling influence their thinking?” To answer this question, I used my analysis of the interview questions. Only one of the 21 participants used the manipulatives without my suggestion. In total, 12 interviewees utilized the manipulatives in at least one question.

The following conversation is an excerpt from an interview with Kristin, a junior from Wilks High School. We are discussing the question found in Figure 4.11.

![Figure 4.11. Interview Question D](image)
Kristin (K): [After working on the problem for several minutes] I just took a wild guess. I don’t know how to get the answer.

Researcher (R): All right. Well you tell me how you got that wild guess.

K: Uhhhh. I don’t know. I just…guessed.

[The discussion proceeds in a similar fashion for several more minutes]

R: OK. Let’s get the blocks out [referring to the connect blocks].

R: You guessed that one-third is two boxes. So, here (pulling out two blocks) is one-third. So what would another third look like? How many boxes would I have in another third?

K: Two.

R: (Taking out two more blocks) So, how many thirds is that?

K: Two thirds.

R: And how many boxes is it?

K: Four.

R: So what do you think about your answer?

K: It’s right? (laughing)

R: Do you mean “it’s right?” or “it’s right!”?

K: (laughing) It’s right!

R: And now, I want to know the whole. How many more thirds do I need to make a whole?

K: One more (taking out two blocks).

R: And how many thirds do we have now?

K: One whole.

R: And, how many boxes?
The following is an excerpt from an interview with Aisha, from Bishop High School. We are discussing the question found in Figure 4.12.

In Chester, the demand for apartments was analyzed and it was determined that to meet the needs of the community, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose the builder is planning to build a large apartment complex containing between 35 and 45 apartments. Exactly how many apartments should he or she build to meet this regulation? How many one-bedroom, two-bedroom, and three-bedroom apartments will there be?

Figure 4.12. Interview Question H

Aisha (A): So far I drew out how many apartments in each thing.

Researcher (R): OK. Sounds good.

A: Now I’m trying to figure out how many apartments so they should meet the regulation.

R: All right.

A: I’m not sure if this is supposed to be a paragraph or not.

R: OK. Well let’s just solve it however you want to. These problems have lots of different ways we can look at them.

A: [After a long pause] I’m not really sure about this one.

R: OK. Let’s just talk about it. So, tell me what you drew and what that means.

A: For the one-bedroom they had three apartments so I had drew that (see Figure 4.13).

For the two-bedrooms I had drew four because they had four apartments. And the three-bedroom, they had one apartment.
R: Right. So if they just build what you have drawn, are they going to meet their requirements?
A: No.
R: How come?
A: It’s, it’s not enough.
R: All right. So we have to build more.
A: Uh-huh.
R: Well, we know how we have to build them. You told me (pointing to the paper) how we have to build them. So, what happens if we start building more? What will it look like?
A: (After a long pause) What do you mean?
R: Well, we know that right now we are not meeting our requirement of 35 to 45 apartments, so we have to build more.
A: Uh-huh.
R: But we know that we can’t go and build whatever we feel like. We know how we have to build them. How should we build them?
A: Three one-bedrooms, four two-bedrooms, and one three-bedroom.
R: So let’s say we started to build again. How many apartments should we build?
A: I could probably; I would probably add every four or multiply every four or something like that.
R: You probably could. What do you think you should do?
A: I think I should multiply.
R: OK.
A: [Looks at the paper for a long time and then multiplies the one-bedrooms by three, the two-bedrooms by four, and the three-bedrooms by one as per the picture below.]

A: I’m not sure I’m right.

R: Well, that’s not always the most important thing. Let’s just give it a try.

A: I added them and got 26. But it still don’t get up there. So, how much do you want to get? I still didn’t get it.

R: Do you understand what you are doing?

A: [Just stares at the paper]

R: I have a suggestion. Let’s use our blocks.

R: All right. So, the first time we build, how many one-bedrooms do we build?

A: [Picks up three blocks] Three.

R: Let’s start building two-bedrooms.

A: [Picks up four blocks and then one more block for the three-bedroom apartment]

R: Can this configuration of three, four and one blocks change? Does it have to stay the same?

A: I’m not sure.

R: Well, let’s look at the problem.

A: [Reading from the problem] “Every time they build three one-bedroom apartments, they should build four two-bedroom apartments and one three-bedroom apartment.”

R: So, if I just want to build sixteen two-bedroom apartments, can I do that?

A: No.

R: Why can’t I do that?
A: Every time you change something to the first one, you can’t do it without changing something to the other two.

R: Exactly, because otherwise I might not have any people to live in my apartments. Then, that’s a big waste of money. But, right now I don’t have enough apartments. There’s going to be too many people and nowhere for them to live. So, I need to start building again.

A: You can’t do one without doing the other two. You can add for it?

R: What do you think?

A: [Takes three blocks, then four blocks, then one block] Now I have six of these and eight of these and two of these.

R: So, now how many apartments have I built total?

A: Can I write down what we just now said?

R: Oh, sure.

A: You have sixteen.

R: So, is that enough?

A: No.

R: So, what should we do now?

A: Add some more. [Takes three block, then four blocks, then one block] You have twenty-four.

R: Do you want to stop?

A: Not yet.

Aisha continued to take out blocks until her apartment total reached 40. Then she was able to count the number of each apartment type.
The students in my study definitely benefited from their use of manipulatives. Thinking about proportional situations abstractly proved too much for many of the participants. Once they were able to use concrete manipulatives to model the situation physically (along with a little help from me) they correctly solved the problem. Many interviewees appeared to be operating largely as concrete thinkers. Physically representing the interview questions was essential in their being able to understand, and ultimately solve, the mathematical problems. The implications for teachers are significant and are discussed in greater detail in Chapter 5.

In summary, two of the 21 students who participated in the in-depth interviews were classified as “advanced” proportional thinkers. Also, students were largely unable to recognize the multiplicative nature of the interview questions. All but two students did not realize the “cross multiply and divide” algorithm would be an appropriate choice to expedite the solutions of these problems. Further, 12 students used manipulatives in order to answer at least one question. In all cases, the use of those manipulatives, along with
minimal guidance, helped the students make sense of the problem and move closer to a solution. Implications of these findings are discussed in Chapter 5.
CHAPTER 5

CONCLUSIONS

In this chapter the purpose, conclusions, limitations, and implications of my study are discussed. The purpose section contains the reasoning for the study. I briefly restate my findings in the conclusions section of this chapter. I critique this study in the limitations section and suggest modifications for future research. The final section of this chapter explores the implications of this study for the mathematics education community.

Purpose of Study

Proportional thinking plays a crucial role in a student’s mathematical development, both inside and outside the classroom. It is suggested by the National Council of Teachers of Mathematics (NCTM) that direct proportional reasoning instruction take place in grades 5 through 8, with those reasoning skills then used extensively throughout high school (1989). The NCTM (2000) stated that proportional reasoning abilities are necessary in understanding many high school mathematics topics such as linear equations, rates, rational numbers and expressions, similar figures, and area and volume relationships. Further, Slovin (2000) asserted that proportional reasoning “lays the foundation” for algebra (p. 58). Additionally, many everyday tasks require the ability to reason proportionally. This study sought to discover if rural, low-income high school students actually understood, and were able to apply, proportional relationships.
Conclusions

One specific objective of this research was to determine if high school students from rural, low-income areas demonstrated advanced proportional reasoning abilities. The proportional reasoning problems were organized into four constructs: associated sets, part-part-whole, growth, and well-chunked measures. During one-hour interviews, the students were given a total of eight questions; two from each construct. To qualify as an “advanced” proportional thinker, a student needed to solve both questions correctly in at least three of the four constructs. Two out of the twenty-one interviewees (9.5%) received an “advanced” designation.

This study was also designed to determine if students conceptually understood the mathematics behind solving proportions using the “cross multiply and divide” algorithm. Surprisingly, only two students correctly solved at least one of the interview questions using the “cross multiply and divide” algorithm. Since the interview questions did not have the same structure as many textbook problems, students did not recognize the proportional nature of these questions. Therefore, only students with strong proportional reasoning abilities were able to both recognize the proportional nature of the questions and utilize the “cross multiply and divide” algorithm to expedite the solution process.

Third, I sought to determine if using manipulatives aided the students in finding a correct solution to the interview questions. Many interview participants appeared to be concrete thinkers. Visualizing the proportional problems abstractly did not lead many students to solutions of the interview questions. Once the interviewees were able concretely model the proportional situations using manipulatives, they were better able to make sense of the questions and move towards a solution.
Limitations

Population

The results of this study are limited to the specific population of students who participated in the Promising Youth program at Southeastern University during Summer 2005. The students were primarily African-American and were all from low-income, mostly rural high schools in a particular southeastern state. While it is my belief that results would be similar if this study were conducted on other populations, at this point, the scope of the findings is limited. To obtain a broader look at proportional thinking, more research is needed.

Future research analyzing the proportional thinking of late adolescents and adults needs to be conducted. To obtain a clearer picture of proportional understanding as a whole, the populations should include students from different areas of the country, different ethnicities, varying degrees of wealth, and both city and suburban environments. Studies on adults should include those in both blue- and white-collar jobs. Another interesting study could investigate the relationship between job type and level of proportional thinking. Do jobs that require the use of those thinking skills influence a person’s ability to reason effectively?

Time Restriction

This study was conducted during the allotted “mathematics time” in the Promising Youth program. Before conducting the research, the interview questions were tested on a variety of people, including both an elementary school student and a middle school student. From this test, I determined that an interview time of one hour would be appropriate to complete all eight questions. The research design was based on one-hour
interview times. Unfortunately, once the research began, I soon realized that more time for the interviews was needed. Many participants were not able to complete all eight questions in the allotted time. The students also had limited time at Southeastern University so continuing the interviews at a later date was impossible. Each question that was not attempted by an interviewee received a rating of N/A. Since the students were unable to attempt all eight questions because they struggled on earlier problems, the N/A rating was considered not advanced. Nevertheless, I believe I garnered a good picture of the students’ overall proportional thinking abilities with these interviews.

To ensure a suitable quantity of data, the research design sought to maximize the number of student interviews. Twenty-one interviews were completed during the summer of 2005. In the future, it would be beneficial to allow more time for these in-depth interviews even if that meant sacrificing the quantity of the interviewees. An accurate representation of the participants’ proportional reasoning abilities was obtained during this study, but, in future research, students need adequate time to explore each of the eight interview questions fully.

*Question Design*

All interview questions utilized in this study were adapted from research-based work published by Lamon (1993b, 1994, 1999) and Marshall (1993). The final interview questions were tested by several different individuals prior to the study. None of the individuals who tested the questions exhibited confusion about the apartment question regarding number of apartments versus bedrooms in the apartments. During the interviews, some students were puzzled by the apartment question, confusing the number of apartments with the number of bedrooms. This situation was quickly remedied through
discussion and pictorial representations. If I were to conduct further research, I would reword the question to eliminate that possible confusion. Instead of using one-, two-, and three-bedroom apartments, I would refer to the apartments as red, blue, and green.

During the first interview, I discovered that one of the Associated Sets questions, the candle question, in its original form, was too difficult for the students to complete, regardless of their proportional reasoning abilities. After allowing the students to struggle with the problem, I offered assistance by assigning an initial length of 18 inches for each candle. If this question were used again, the researcher should be prepared to offer similar assistance. Using a specific example of a general problem is a good mathematical strategy that can go beyond this study.

The original Associated Sets candle question was worded very generally: “Two candles are made of different kinds of wax. They are of equal length and they are lighted at the same time, but one candle takes 9 hours to burn out and the other takes 6 hours to burn out. After how much time will the slower-burning candle be twice as long as the faster-burning candle?” I suggested the 18 inch solution strategy to impose specificity on an otherwise abstract question. I realized that the students had difficulty thinking abstractly and that difficulty would compromise their ability to solve this problem. I would not have been able to evaluate their proportional reasoning abilities because they were not able to get past the generalization. A student who had the ability to think abstractly, with a strong understanding of proportional relationships, may have been able to reason through the original problem, without the help, to find a solution.

Prior to conducting the research, I gave this problem to several people who were all able to make significant progress toward a solution. I did not realize, though, that I
would have so few abstract-thinkers participating in this study. I believe that the two students who were classified as “advanced” in the study were the only participants able to think in abstractions. Therefore, I would not expect the remaining interviewees to solve this problem given the original wording. After receiving the 18 inch assistance, the students had the option of modeling the situation with pictures or connect blocks. Also, after the help, the amount of candle that burned every hour could be calculated numerically as 3 inches per hour and 2 inches per hour for the faster and slower burning candles, respectively. Before the hint, the amount of candle that burned every hour could not be represented numerically, but, rather, as one-sixth and one-ninth of the candle each hour. All students with strong proportional reasoning abilities should be able to answer the question after the hint.

Throughout my research, I found that students appeared more able to solve problems that were able to be modeled using manipulatives or pictures. Assigning a length to the candles allowed the students to model the situation with connect blocks, pictures, or other manipulatives. Without the specificity suggestion, the interviewees demonstrated no ability to construct a solution strategy. The students were unwilling, or unable, to make any attempt to move toward an answer. Once the hint was given, those same students felt comfortable enough to take a risk and suggest ways to find the amount of candle burned each hour, most with the aid of manipulatives.

It was interesting to witness the immediate change in the attitude of the student towards the mathematics problem, depending on the context of that problem. When the students were working in groups, in the whole-class situation, and working on problems like the lemonade and capture-recapture activities (Appendix A), they were interested and
alert. The students did not appear afraid or overwhelmed. The same attitude was apparent
during the interviews when the students either knew how to solve a problem or were
using manipulatives to move toward a solution. A change occurred when the students
encountered the speeding ticket and candle questions. They did not understand the
questions and could not move toward a solution. After the candle hint, the students
attacked the problem with a renewed energy.

When presented with unfamiliar, abstract mathematical ideas, the students
responded negatively. The same principle applies in a classroom environment. Presenting
students with new abstract concepts may produce anxiety or overwhelm them. Allowing
the students to become familiar with the ideas, through familiar contexts, may ease a
students’ mind and allow more learning to take place. Once a student is ready, moving to
the abstract concept is part of the natural progression. This move from concrete to
abstract is supported by the National Council of Teachers of Mathematics (2000). This
idea is further supported by research conducted by Cramer, Behr, Post, and Lesh (1997).
Working with the Rational Number Project, they developed a curriculum based on the
following principles:

1. Children learn best through active involvement with multiple concrete models
2. Physical aids are just one component in the acquisition of concepts: verbal,
pictorial, symbolic and real world representations also are important
3. Children should have opportunities to talk together and with their teacher about
   mathematical ideas
4. Curriculum must focus on the development of conceptual knowledge prior to
   formal work with symbols and algorithms (p. 2).
They developed an entire curriculum devoted to starting with concrete models and move toward abstraction. A subsequent section of this chapter offers the interviewees’ experience with manipulatives.

Too often, in elementary school through high school, teachers start with the abstract concept. For instance, in a high school mathematics classroom, during a lesson on absolute value, a teacher might write the following:

If $p < 0$, then $|p| = -p$

If $p > 0$, then $|p| = p$.

The follow-up question to that generic rule might be: “What is $|-2|$?” Even to many abstract-thinking adults, this representation is not readily understandable. To a concrete-thinking student, this representation is confusing, meaningless, and overwhelming and might prompt that student to tune out the rest of the lesson. If that overwhelming experience happens often enough, a student may begin to develop a dislike for math altogether. Certainly, making a student memorize that particular absolute value rule leaves little room for understanding or for the creation of meaning.

An alternate way of exploring the idea of absolute value would be through a distance representation: the absolute value measures a number’s distance from zero. Ask the students if distances could ever be negative. Allow them to develop their own rules. Approaching the concept from a more contextual perspective, and letting them develop the abstract rule, allows students to create meaning and increases the chance that students will retain the information and be able to utilize it in the future.
Although the interview questions used in this study were based on research-supported proportional reasoning problems, they could be further delineated to better evaluate the thinking skills of the interviewees. Before conducting further research, some questions should be altered based on the data in this study. Doing so partly decreases the chance of ambiguity or vagueness in the questions. From the research, it is apparent that many students were not able to solve many of the problems, even though these were based on middle school curriculum.

Implications

Teachers

Recognizing multiplicative relationships: When to use “cross multiply and divide”

Many students in this study were unable to recognize the underlying multiplicative relationships in the interview questions. For example, 15 of the 21 interviewees initially assigned an additive relationship to the cookie recipe question. Since there was ½ of a cup less of the milk, they assumed that they would also need to subtract ½ from both the tablespoon of butter and the ½ cup of sugar, even though quantities and units were different. Anticipating this student response, I created the original recipe calling for only ½ of a cup of sugar, yielding no sugar after the subtraction. It was only after deducing that they needed no sugar that the students rethought their strategy. Interestingly, two of the students did not alter their responses; one student even stated that he “just wouldn’t use any sugar in these cookies.”

The students completed a similar activity, in a whole-class situation, working in groups, during the math class time of Promising Youth. In this activity, found in Appendix A, the students were given two different recipes for making fruit punch from a
mix; one required 5 cups of water for every 2 tablespoons of mix and one required 7 cups of water for every 4 tablespoons of mix. During the activity the students used actual measuring cups, mix, water, and pitchers to make the different batches of fruit punch. The students were prompted to answer questions such as “How many cups of fruit punch could be made, using recipe 1, if you only had one tablespoon of mix?” and “How much mix would you need, using recipe 1, if you only wanted one cup of fruit punch?”

During the activity the students were able to answer the questions correctly. The students, in their groups, demonstrated the ability to recognize the multiplicative nature of the activity. Two of the major differences between the interview situation and the fruit punch activity were that in the activity the students worked in a group setting and actively participated in making fruit punch. Through their interactions with each other, as well as with the fruit punch, the students were able to achieve more success than working alone.

Using discussion as part of the learning process for fractions, ratios, and proportions is supported (Cramer, Behr, Lesh, & Post, 1997). Perhaps if the interviewees actually worked with the milk, sugar, and flour, they would have initially recognized the appropriate manner in which to reduce the other ingredients. More manipulatives discussion follows in the next section.

In the cookie interview question, those who eventually recognized that the recipe reduction relationship was not additive, but, rather, multiplicative, had a very difficult time solving the problem. The students became particularly flustered when faced with the task of taking away ¼ of, or 25% from, the ½ cup of sugar. None of the students recognized that \( \frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \) signified that they should use \( \frac{1}{8} \) less cup of sugar, or
\[ \frac{1}{2} - \frac{1}{8} = \frac{3}{8} \] cups of sugar. An alternative solution, still not using proportions, would have been to take a 25% reduction on the sugar: 

\[ 0.5 \times 0.25 = \frac{0.25}{2} = 0.125 \] cups less of the sugar, giving 0.375 cups of sugar. The unit of 0.375 cups of sugar is not easily obtained in real life, without converting it to a fraction, since most measuring cups are labeled in fractional, not decimal units. The decimal solution was included because data gathered during the study, and information obtained from teaching experiences, suggested that many students are uncomfortable with fractions and will convert them to decimals at every available opportunity.

A third, and perhaps most straightforward, way to solve this problem was through a proportion. For example, consider the proportion 

\[ \frac{1\text{ cups of milk}}{2\text{ cups of milk}} = \frac{x\text{ cups of sugar}}{\frac{1}{2}\text{ cup of sugar}}. \]

If students were uncomfortable with fractions, they could use 1.5 cups of milk and 0.5 cups of sugar in the proportion. Using the cross multiply and divide algorithm immediately yields an answer.

Only two out of the 21 students recognized that a proportion could be used to solve the problem, although all 21 were familiar with setting up proportions and solving with the cross multiply and divide algorithm as that is part of the Algebra 1 curriculum. This inability to recognize the proportional relationship suggests that these students, unless they are given a proportion and told to “solve for x,” do not have an understanding of when to utilize a proportion and solve it. Further, when “solving for x,” they are robotically following a pre-determined set of steps and do not understand why the procedure works. Therefore, when they are given a familiar situation such as

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and asked to “solve for x,” they are likely using the algorithm without understanding. However, given a contextual situation that could be modeled through the use of a proportion, the students fail to recognize the connection. These students do not truly understand proportional relationships. Carpenter (1986) suggested that introducing an algorithm before developing a conceptual understanding creates a curriculum that is difficult for students to master or appreciate.

Why teach students an algorithm if they cannot apply it to a real-life situation? It seems to me that the time spent over a span of four or five years devoted to teaching the cross multiply and divide algorithm is mostly wasted time if all students can do is solve for x in problems similar to \( \frac{35}{250} = \frac{x}{900} \). The curricular focus should not be teaching an algorithm without understanding. Not understanding why or when to use an algorithm is, at best, not helpful. In fact, it can be damaging as it may further the distance between “school math” and real life.

Perhaps, instead, we should take a more Deweyan (1997) approach and put the child in a real-life situation where a problem situation presents itself. For example, a third grade class is having an end of year party and the teacher assigns different groups of students to different tasks. One group is in charge of making the lemonade from a mix. The directions on the package state that three scoops should be mixed with 2 quarts of water. The classroom only has 1 gallon containers. The students have the mix and the gallon containers and need to determine how much mix to put in those containers. The teacher should let the students develop a strategy for solving their problem and even let them “test” the taste of the mixed juice. After this and several other mathematically
equivalent experiences, a teacher could move toward the abstract understanding and introduce the idea of conversions and proportions. Students would then have a familiarity with applying the algorithm rather than just memorizing the steps out of context. Introducing “algorithms” amidst a situation relevant to a student’s life creates an opportunity to connect that algorithm to an application.

Concrete manipulatives

During this study, I found the use of concrete manipulative positively affected a student’s ability to both make sense of the problem situations and find an appropriate solution. In the Part-Part-Whole question, where a student was presented with four squares comprising two-thirds of the whole, many were unable to move towards a correct answer. I intentionally used pictures of the squares on the question page, rather than the words “four squares are two-thirds” to determine the impact of those pictures on student understanding. Five students drew on the question sheet, circling two of the four squares and labeling that one-third. The other students did not exhibit behavior to suggest that the pictures were an additional source of understanding. Once they were able to physically use connect blocks to represent those squares, most were able to complete the problem successfully. This result suggests that manipulatives are important, particularly when addressing a topic such as fractions, and this finding is consistent with suggestions from the National Council of Teachers of Mathematics (2000). Further, Ball (1993) asserted that “in order to help students develop mathematical understanding and power, the teacher must select and construct models, examples, stories, illustrations, and problems that can foster students’ mathematical development” (p. 159). Most students participating in this study were not formal operational thinkers and, thus, required concrete and visual
approaches before moving towards abstractions. Many students’ “gut-instinct” was to say that the “whole” was made up of six squares, but were unable to explain their thought process. The physical act of moving two additional connect blocks to the original four to make the “whole” allowed the students to take ownership of their mathematical ideas.

During the interviews, students often asked if their answers were correct. I usually responded by asking them if they thought they were correct. At first, the interviewees were caught by surprise and did not know how to respond. Students are comfortable in the familiar relationship of the teacher as the owner and dispenser of knowledge and the student as the receiver of that knowledge. In the classroom, finding the “correct” answer to a problem and getting confirmation from the teacher should not be the final step of a mathematics problem. Teachers need to delve more deeply and ask questions such as: “Does the student know why the answer is correct?”, “Could the student apply this principle to a different problem?”, and “Is the student just following a set of steps without understanding or does he understand the underlying concepts?” Allowing students to take control of their learning leads to deeper understanding.

Understanding “speed”

During the course of this study, I unintentionally discovered that many students do not have an understanding of rates and, in particular, did not understand the common rate of miles per hour. During the design of the study, I made an assumption that students would be able to utilize their understanding of both miles per hour and average speed to solve the speeding ticket question. When asked to describe the term “miles per hour” in their own words, most students faltered. I asked four of the students the following question, “If it took me 2 hours to drive 120 miles, how fast was I driving, on average?”
They did not know the answer. I followed with “If I drove 60 miles in 1 hour, how fast was I driving, on average.” Again, surprisingly, the students were not able to give a speed. These interviewees were unable to use the relationship between distance, time, and rate (speed) to solve the problem. Lamon (1993b) asserted that “a student can have a strong sense of the size and nature of quantities such as speed and may use the appropriate vocabulary (e.g., miles per hour) accurately without being aware of the relations between the numbers that compose those ratios” (p. 58). This study confirmed this conclusion.

In a classroom situation, teachers must attempt to evaluate a student’s prior knowledge and use that information when preparing lessons, asking questions, and pacing their instruction. If a teacher has not accurately assessed that knowledge, it can be detrimental to student learning. I realized, through this process, that being familiar with a concept and understanding that concept are two very different things; familiarity does not imply understanding. This discrepancy has strong implications for the classroom. If students do not have a conceptual understanding of a mathematical idea or principle then it is difficult to build upon that idea. I see this type of mathematical teaching as building a house on a foundation made of twigs. Having a strong knowledge base is necessary for a student to grow mathematically.

Curriculum

Proportional semantic types

Susan Lamon (1993b) conducted research with sixth grade students investigating proportional thinking skills in the same four constructs as this research. She did not find “apparent transfer of knowledge” between the four types. Students exhibited various
levels of success when attempting to solve problems in the different constructs. Associated sets problems were often solved by formal proportional means, whereas, part-part-whole problems were most often solved through the less sophisticated building-up strategy without the use of proportional reasoning. Lamon (1993b) suggested that students are most comfortable with associated sets problems, and can model them formally, because they most closely mimic those problems used in classrooms. Researchers (Cramer, Post, and Currier (1993); Lamon (1993b)) generally agreed that growth problems are the most difficult for students to understand and solve. Furthermore, Lamon (1993b) suggested that students have more difficulty with the well-chunked measures and growth constructs because they are more difficult to model using physical representations.

My findings support those found by Lamon (1993b). Students were more able to solve the associated sets and part-part-whole problems than the well-chunked measures and growth problems. Many failed to recognize the multiplicative nature of the growth problems. Further, as discussed in detail, these students were not able to utilize the information given regarding “speed.” Perhaps proportional thinking problems should be explored in a specific order. The associated sets type problems are most familiar to students entering high school. They appear most comfortable recognizing the multiplicative relationship between two objects (such as comparing people to pizzas). Using those types of problems as a springboard, teachers could then introduce part-part-whole questions. These questions were often solved using build-up strategies. Teachers could let the students explore the problems using whatever method seems most comfortable and not stop with the correct solution. They could push the students to think
more deeply about the question and compare the solutions of the associated sets problems with the part-part-whole problems. Then challenge the students to find the proportional connection between the two sets of problems and they could help the students recognize the proportional nature of the questions types and that both can be solved used proportions.

Only after students are comfortable and successful in the associated sets and part-part-whole questions should the more difficult, and more abstract, well-chunked measures and growth problems be introduced by teachers. The associated sets questions lay the foundation for future proportional thinking. As with the part-part-whole questions, teachers could connect the well-chunked measures and growth questions back to the associated sets and help the students recognize that an invariant relationship between two objects can be seen as a rate. Together, they could explore the meanings behind such common rates as miles per hour and price per pound. A more specific discussion of well-chunked measures and growth problems follows in the next section.

Proportional problems in Algebra 1 and Geometry

The results of this study suggest that many students who have completed Algebra 1 do not have strong proportional reasoning abilities. The NCTM (1989) suggested that proportional reasoning instruction occur when the student is in grades five through grades eight. Since it is such a pivotal thinking skill, perhaps it could be further explored during Algebra 1 and Geometry.

Linear relationships, equations, functions, and inequalities are all crucial topics in the Algebra 1 curriculum. Exploring these relationships through graphs, charts, and tables before the formal introduction of the “function” can strengthen proportional thinking. For
example, suppose a student wanted to get a cellular telephone. He could choose a plan that did not require a long-term contract and pay per minute of cell phone use. The students could explore the relationship between minutes used and monthly cell phone bill. Alternatively, he could sign a year-long contract which would cost $35.00 a month. The students could compare the yearly cost of each phone given a certain number of minutes used. Which plan would be more economical given the number of minutes he might use? This type of activity would directly strengthen reasoning skills in the associated sets construct. Furthermore, this type of exploration, if done repeatedly, would lead directly into linear functions.

This research study revealed that students struggle with problems in the well-chunked measures construct. Specifically, students seemed unable to understand or utilize rates. If students do not understand the concept of a rate then truly understanding the idea of slope becomes impossible. Again, beginning with a contextual situation, like the cell phone example, might make the abstract concepts of a slope more meaningful. Teachers could allow students to explore the idea of “cost per minute” or “cost per month.” and introduce the idea that those are rates. After sufficient examples, relate a rate to the slope of a line. During the interviews, as discussed previously, I found that students used the term “miles per hour” very casually, but had little idea how that measured speed. Algebra 1 is an excellent time to explore rates and their relationship to slope. It is imperative that students are required to explain what the slope of a line means, as opposed to just finding the numerical value or writing the equation of a line in slope-intercept form. Only then will they strengthen their proportional reasoning abilities in the well-chunked measures construct.
Well-chunked measure problems are also difficult because they are hard to model. Teachers could use motion sensors or probes to allow students to visually see the movement of an object. Then students would have a representation of movement per second.

The participant in this research study also exhibited considerable weakness on the problems in the growth construct. Lamon (1993b) and Langrall and Swafford (2000) found that students were unable to recognize the multiplicative nature of the growth problems. Lamon (1993b) worked with sixth grade students and Langrall and Swafford (2000) studied middle-school aged children. Neither of the populations studied completed Algebra 1 during the time of the investigation. Every student in my study had completed Algebra 1 and many had successfully passed Geometry also; unfortunately, most of the interviewees were still unable to recognize the multiplicative nature of the growth problems. Geometry is a mathematical area that is more easily understood through the use of manipulatives. Also, similarity is a key concept in the study of Geometry. In order to strengthen the students reasoning abilities about growth problems, within the context of geometry, an activity such as the following could be used:
The pieces below may be cut out and then reassembled to make the square.

Call the original square \( \Box GILO \).

1. Make a new set of all six pieces so that:
   The segment that measures 4 cm on the original square should measure 12 cm of the new version.
   Each person can work on different pieces. When everyone is finished, you should be able to put the new pieces together to make a square. The new square will be called \( \Box G_1I_1L_1O_1 \).

2. Make a new set of all six pieces so that:
   The segment that measures 4 cm on the original square should measure 7 cm on the new version.
   Each person can work on different pieces. When everyone is finished, you should be able to put the new pieces together to make a square. The new square will be called \( \Box G_2I_2L_2O_2 \).

3. Write the ratios between:
   a. Side \( \overline{GP} \) to side \( \overline{PO} \)
   b. Side \( \overline{GP} \) to side \( \overline{G_1P_1} \)
c. Side $\overline{GP}$ to side $\overline{G_2P_2}$

4. What is the ratio of the perimeters of $\Box GILO$ to $\Box G_1L_1O_1$? What is the ratio of the perimeters of $\Box GILO$ to $\Box G_2L_2O$?

5. What is the ratio of the areas of $\Box GILO$ to $\Box G_1L_1O_1$? What is the ratio of the areas of $\Box GILO$ to $\Box G_2L_2O$?

6. Why are the perimeter and area ratios different?

7. Write a set of instructions for enlarging and reducing the puzzle pieces. Explain why the procedure worked. Describe any method that did not work, and explain why it did not work.

The students could explore the stretching and compressing of objects, without strictly receiving and memorizing an abstract set of rules. Through similar activities, the students could strengthen their abilities to recognize multiplicative relationships among objects.

Proportional reasoning activities do not need to be limited to late elementary and middle school mathematics classrooms. If students are unable to reason proportionally, the topics need to be addressed in later mathematics. Incorporating activities into both Algebra 1 and Geometry math classes increases the likelihood that students will be able to utilize these thinking skills later in life. As a community college mathematics teacher, I have witnessed students with very weak proportional reasoning abilities. These students were able to complete high school mathematics successfully, but struggled when they got to college. Perhaps the effects of this weak reasoning are not apparent immediately, but surface once the students move to college math. Since the students in this study have all passed Algebra I and many have passed Geometry and even Algebra II, we can only conclude that they may have memorized algorithms sufficiently to survive the class, but it is doubtful that the mathematics has any meaning for them.

Conclusion

During this study I found that high-school students from rural, low income area do not have strong proportional reasoning abilities. Although the NCTM (1989) suggests
that these reasoning skills should be explicitly taught in 5\textsuperscript{th} – 8\textsuperscript{th} grades, perhaps it would be beneficial to continue that instruction in high school. Students need to be able to utilize those thinking skills so that incorporating proportional activities into high-school classrooms could be beneficial. Also, modeling problems using manipulatives, even in high-school, might further strengthen students understanding of proportional problems. Only after repeated experience should students move from concrete representations towards abstract ideas. Maybe then these students will develop a deeper understanding of proportional situations and successfully use those skills in subsequent education and real life.
APPENDICES
Appendix A

Whole – Class Activities
Capture – Recapture

Capture-recapture is a statistical method used to estimate the size of a population. Fish and wildlife management experts, demographers, and scientists use this and other techniques to find the number of people or animals in a region.

13. Discuss the tagging technique.

14. Locate the classroom population of “bean fish”.

15. Each group will “capture” approximately 20 – 30 bean fish and tag them (you can use the markers provided).

   a. My team tagged this many bean fish:

16. Each group will place the “tagged” bean fish back into the large population.

   a. The total number of bean fish tagged by the class:

17. In your group devise a plan to use the “tagged” fish to estimate the entire population of bean fish. Describe the procedure below.
18. Share the plans as a class and decide on the most effective plan.

19. Each group takes three samples of the large population. Estimate the size of the entire bean fish population using the class plan.
   
a. Sample 1

   Population estimate:

b. Sample 2

   Population estimate:

c. Sample 3

   Population estimate:

20. Average the three population estimates from your group.

   Average group estimate:

21. Combine the results from each individual group to create a class population estimate.

   Class estimate:
22. How do your estimates compare with the other groups? How do your estimates compare with the class estimate? Why might these numbers be different?

23. Suppose you wanted to estimate the population of deer in your county. Describe the procedure you would use (without capturing all of the deer).

24. Why does your procedure work?
Lemonade or Fruit Punch Mix

Below are two recipes for making lemonade (or fruit punch). Recipe 1 calls for 2 tablespoons of mix for each 5 cups of water. Recipe 2 calls for 4 tablespoons of mix for every 7 cups of water. Complete the table to determine the amount of mix and water for the given numbers of pitchers.

<table>
<thead>
<tr>
<th>Pitchers</th>
<th>Recipe 1</th>
<th>Recipe 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mix</td>
<td>Water</td>
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<td></td>
<td>(tablespoons)</td>
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<td>10</td>
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</tr>
</tbody>
</table>
Questions:

1. Which recipe should taste more “lemony” (or “fruity”)? Explain at least two ways the table can help you choose the more lemony recipe.

2. For Recipe 1, how many tablespoons of mix would you need to make 100 cups of lemonade (or fruit punch)? For 1000 cups? Explain how you got your answers.

3. Suppose that you only had 1 tablespoon of mix. How many cups of lemonade (or punch) could you make using Recipe 1? Using Recipe 2? Explain how you got your answers.

On the grid below draw line graphs for the amounts of mix and water for Recipe 1 and Recipe 2. Make sure to label each graph!

Mix versus Water

5. How are the graphs similar? How are they different? Why is one graph “higher” than the other? Why is one graph “steeper” than the other?
6. As the amount of mix increases, do the graphs seem to be further apart or closer together? Explain why this is happening.

7. How can you use the comparison of the two graphs to determine which recipe is more lemony (or fruity)?

Now, decide which recipe you want to use and make a pitcher of juice for your group. Good Work!!
Name ______________________________
School ______________________________
Grade Next Year ______________________

Ready, Set, Go

Roll the car backwards and let it go. Determine a method for finding the average speed of
the car in centimeters per second (cm/sec). You can use the car, the stopwatch, and the
measuring tape.

Describe the method below.

What is the car’s average speed?

Using the information above, predict how long it will take the car to travel 70 cm.

Using the measuring tape and stopwatch, determine how long it takes the car to travel 70
cm.

Activity created by:  Dr. Robert Horton and Kirsten Bernasconi
Were the predicted time and the actual time different for the 70 cm drive? If so, why do you think they were different?

Predict how long it would take the car (if it could) to travel 1 kilometer.  
[HINT: 1km = 100,000cm]

Measure the length of the car in centimeters.  
Determine the average speed of the car in car lengths per second.

A regulation NASCAR Busch Series car must adhere to the following guidelines (found at http://www.jayski.com/pages/diff.htm):

Length 203.5 inches  
Width 74.5 inches

Suppose the NASCAR racecar traveled at the same speed at the Tiny Tunerz® (they traveled at the same car length per second speed).  
What would the NASCAR racecar’s average speed be in feet per second?

What would the NASCAR racecar’s average speed be in miles per hour?  
[HINT: 1 mile = 5280 feet]
The winner of the May 28th Carquest Auto Parts 300 race at Lowe’s motor speedway was Kyle Busch. His average speed during that race was 117.968 mph (found at http://www.lowesmotorspeedway.com/news_photos/news/477554.html) or 10.203 car lengths per second.

If our Tiny Tunerz® car went at the same speed, how long should it take the car to go 70 cm?

Is the toy realistic? Why or why not?
Make a New Puzzle

The pieces below may be cut out and then reassembled to make the square.

Call the original square □GILO.
2. Make a new set of all six pieces so that:

The segment that measures 4 cm on the original square should measure 12 cm of the new version.

Each person can work on different pieces. When everyone is finished, you should be able to put the new pieces together to make a square. The new square will be called $\square G_1I_1L_1O_1$.

3. Make a new set of all six pieces so that:

The segment that measures 4 cm on the original square should measure 7 cm on the new version.

Each person can work on different pieces. When everyone is finished, you should be able to put the new pieces together to make a square. The new square will be called $\square G_2I_2L_2O_2$.

4. Write the ratios between:
   
   a. Side $\overline{GP}$ to side $\overline{PO}$
   
   b. Side $\overline{GP}$ to side $\overline{G_1P_1}$
   
   c. Side $\overline{GP}$ to side $\overline{G_2P_2}$

5. What is the ratio of the perimeters of $\square GILO$ to $\square G_1I_1L_1O_1$? What is the ratio of the perimeters of $\square GILO$ to $\square G_2I_2L_2O$?

6. What is the ratio of the areas of $\square GILO$ to $\square G_1I_1L_1O_1$? What is the ratio of the areas of $\square GILO$ to $\square G_2I_2L_2O$?

7. Why are the perimeter and area ratios different?

Activity adapted from:  
*Classroom Activities for Making Sense of Fractions, Ratios, and Proportions*
8. Write a set of instructions for enlarging and reducing the puzzle pieces. Describe any method that did not work, and explain why it did not work.
Appendix B

Interview Questions
These two rectangles are the same shape.

Find the height of the larger rectangle.
You get the following “special” offer in the mail for a monthly magazine subscription:

A. a 6-month subscription for 2 payments of $6.00 each – OR –
B. a 9-month subscription for 3 payments of $6.00 each

1. Is option A a better deal than option B? Why or why not?

2. The magazine also offers a 12-month subscription for 4 payments of $5.00 each. Is it a better deal to subscribe for 12 months? Why or why not?
Would a girl or a boy get more pizza?

The Girls

The Boys
If $\frac{2}{3}$ is $\frac{2}{3}$, draw the whole.

If $\frac{2}{3}$ is $\frac{2}{3}$, draw $\frac{1}{3}$

If $\frac{2}{3}$ is $\frac{2}{3}$, draw $1\frac{1}{3}$

If $\frac{2}{3}$ is $\frac{2}{3}$, draw $1\frac{1}{2}$
I am making cookies and my recipe calls for 2 cups of milk, 1 tablespoon of butter, and $\frac{1}{2}$ cup of sugar. Unfortunately, I only have $1 \frac{1}{2}$ cups of milk. I need to use less butter and sugar so that the cookies still taste good. How much butter and sugar should I use?
In Clemson, the demand for apartments was analyzed and it was determined that to meet the needs of the community, builders would be required to build apartments in the following way: Every time they build 3 one-bedroom apartments, they should build 4 two-bedroom apartments and 1 three-bedroom apartment. Suppose the builder is planning to build a large apartment complex containing between 35 and 45 apartments. Exactly how many apartments should he or she build to meet this regulation? How many one-bedroom, two-bedroom, and three-bedroom apartments will there be?
Charlie was driving on a highway containing a speed trap. The speed trap works in the following way: a camera records the time each car drives past a particular point on the highway. Another camera records the time each car passes a second point 10 miles down the road. The average speed of the car over that 10 mile stretch is then automatically calculated. The driver of a car is issued a speeding ticket if his average speed is above the 55 mph speed limit.

Just when Charlie entered the speed trap he noticed a police car on the side of the road. The police officer pulled out behind Charlie. Not wanting to get a ticket, Charlie immediately slowed down and drove a constant 40 miles per hour. After five miles the police officer gave up and stopped following Charlie. Since the police officer turned around, Charlie began driving faster. Charlie passed the second checkpoint on the highway and was pulled over. The computer calculated that his average speed was 53 miles per hour for the entire 10 mile stretch of road. Surprisingly, the police officer who pulled Charlie over gave him a ticket for driving 79 miles per hour. Charlie went to court. How did the officer prove that Charlie was guilty?
Two candles are made of different kinds of wax. They are of equal length and they are lighted at the same time, but one candle takes 9 hours to burn out and the other takes 6 hours to burn out. After how much time will the slower-burning candle be twice as long as the faster-burning candle?
Appendix C

Mathematics Pre – Assessment
How do you feel about math? Circle the most appropriate response.

1) I have a lot of confidence when it comes to math.
   Agree                    Disagree                    Not sure

2) I’m no good in math.
   Agree                    Disagree                    Not sure

3) I am the type that does well in math.
   Agree                    Disagree                    Not sure

4) For some reason even though I study, math seems unusually hard for me.
   Agree                    Disagree                    Not sure

5) Boys are better at math than girls.
   Agree                    Disagree                    Not sure

6) If I had good grades in math, I would try to hide it.
   Agree                    Disagree                    Not sure

7) It would make people like me less if I were a really good math student.
   Agree                    Disagree                    Not sure

8) My teachers think I’m the kind of person who could do well in math.
   Agree                    Disagree                    Not sure

9) I have found it hard to win the respect of math teachers.
   Agree                    Disagree                    Not sure
10) My teachers think advanced math is a waste of time for me.
   Agree  Disagree  Not sure

11) It’s less important for girls to learn math than for boys.
   Agree  Disagree  Not sure

12) Math is a worthwhile and necessary subject.
   Agree  Disagree  Not sure

13) Math will not be important to me in my life’s work.
   Agree  Disagree  Not sure

14) I expect to have little use for math when I get out of school.
   Agree  Disagree  Not sure

15) Math usually makes me feel uncomfortable and nervous.
   Agree  Disagree  Not sure

16) I get a sinking feeling when I think of trying hard math problems.
   Agree  Disagree  Not sure

17) My mind goes blank and I am unable to think clearly when working math.
   Agree  Disagree  Not sure

18) Math is enjoyable and stimulating to me.
   Agree  Disagree  Not sure

19) I am challenged by math problems I can’t understand immediately.
   Agree  Disagree  Not sure

20) Figuring out math problems appeals to me.
   Agree  Disagree  Not sure
21) I would rather be told the solution to a problem than have to work it out for myself.
   Agree  Disagree  Not sure

22) I do as little work in math as possible.
   Agree  Disagree  Not sure

**SOME PRELIMINARY PROBLEMS**

Solve the following. Show any work you do, clearly indicating how you have solved the problem. You may use a calculator if you wish.

1) An entertainment system that normally sells for $650.00 is on sale this week for 30% off the regular price. How much would you save if you bought the system on sale?

2) Simplify using the Order of Operations: \( 3^2 + 5 \times 4 \)

3) Add: \( \frac{1}{2} + \frac{2}{3} \)

4) The neighborhood wants to build a fence around the playground. How much fence do they need if the playground is to be 10.5 yards wide and 12.2 yards long?

5) Find the quotient: \( 9 \div \frac{1}{9} \)

6) Susan can run one mile in 10 minutes. The world record holder, Joseph Kimani ran the 10-kilometer in 27 minutes and 20 seconds. This is approximately 4.409 minutes for each mile. At this rate, how many minutes faster can he run a mile compared to Susan? (Round to the nearest tenth.)
7) It takes Mark 4 hours to cut the grass, but Todd can cut it in 3 hours. About how long would it take them if they work together? Explain your answer.

8) Your 10th grade class is taking a field trip. There are 29 students, and 4 can go in each car. How many cars are needed? Explain your solution.

9) Suppose you had $100 and found an investment that paid 6% annual interest. How much would the investment be worth in 5 years? Show your work.

10) State the next three numbers in the given sequence. 16, 8, 4, 2, ____ , ____ , ____ .
11) Suppose you have $130 to take on a trip and spend $15 each day.
   A. In the grid below, sketch a graph showing the relationship between the number of
      days and the amount of money you have left. Be sure to label your axes.
   B. How long would your money last? How can you tell from the graph?

   [Graph grid]

12) Express $\frac{3}{5}$ as a decimal and as a percent.

13) Round 326,489 to the nearest hundred.
14) Simplify (do the problem): 
   \((-13) - (-21)\)

15) Simplify: 
   \(\frac{3}{4} \times \frac{5}{6}\)

16) On a scale map, 2 cm represents 5 miles. If two cities are drawn 5 cm apart on the map, how far apart are they in reality? Show any work you do.

17) Simplify: 
   \(\frac{24}{-8}\)

18) Simplify: 
   \(-14 + -14\)
SCORING RUBRIC

**Attitudes**

Students can score up to 40 points. Higher scores indicate positive attitudes towards mathematics. Some questions need to be reverse scored. Questions 5 and 11 probe students’ views about gender/math and may be interesting to analyze separately. The following problems must be reverse keyed so that 2 points is awarded for a positive response:

Questions: 2, 4, 6, 7, 9, 10, 13, 14, 15, 16, 17, 21, 22

Positive responses = 2 points
Not Sure = 1 point
Negative responses = 0 points

**Skills**

On “Some Preliminary Problems,” questions 2, 3, 5, 10, 12, 13, 14, 15, 17, and 18 require skills without context. These are to be scored with the following rubric. Ten points is maximum.

1: A correct response is supplied
0: An incorrect or no response is supplied.

**Problems**

Problem solving is assessed with questions 1, 4, 6, 7, 8, 9, 11, and 16. Students can score up to 40 points. Each problem will be scored using the following 6-point rubric.

5: Excellent. A correct solution with logical and clear work is presented.
4: Good. A correct or nearly correct solution is provided, but the work lacks one or more details.
3: Marginally satisfactory. The work shows logical progress toward a correct solution, but a conceptual error led to an incorrect solution or the problem was not finished. Alternatively, this score represents a correct or nearly correct solution, but there is little evidence of the thought process leading to the solution.
2: Weak. Some evidence of conceptual understanding is present, but there are significant misconceptions or errors in thinking.
1: Poor. There is little to no evidence of understanding or of how to obtain a solution.
0: No solution is provided, and there is virtually no work.
REFERENCES


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