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Abstract In this paper we study the equivalence relation on the set of acyclic orientations of a graph Y that arises through source-to-sink conversions. This source-to-sink conversion encodes, e.g. conjugation of Coxeter elements of a Coxeter group. We give a direct proof of a recursion for the number of equivalence classes of this relation for an arbitrary graph Y using edge deletion and edge contraction of non-bridge edges. We conclude by showing how this result may also be obtained through an evaluation of the Tutte polynomial as $T(Y, 1, 0)$, and we provide bijections to two other classes of acyclic orientations that are known to be counted in the same way. A transversal of the set of equivalence classes is given.

1 Introduction

The equivalence relation on the set of acyclic orientations of a graph Y that arises from iteratively changing sources into sinks appears in many areas of mathematics. For example, in the context of Coxeter groups the source-to-sink operation encodes conjugation of Coxeter elements [11], although in general, these conjugacy classes are not fully understood. Additionally, it is closely related to the reflection functor in the representation theory of quivers [6]. It has also been studied in the context of the chip-firing game of Björner, Lovász, and Shor [1]. Moreover, it arises in the characterization of cycle equivalence for a class of discrete dynamical systems [5], which was the original motivation for this work.

In [11], the number of equivalence classes $\kappa(Y)$ for a graph Y was determined for graphs that contain precisely one cycle. The main result of this paper is a novel proof for $\kappa(Y)$ for arbitrary graphs in the form of a recurrence relation involving the edge deletion Y'_e and edge contraction Y''_e of a cycle-edge e in Y. It can be stated as follows.

Theorem 1 *Let e be a cycle-edge of Y. Then*

$$\kappa(Y) = \kappa(Y'_e) + \kappa(Y''_e)\ . \tag{1}$$

Our proof involves a careful consideration of what happens to the κ-equivalence classes of $\mathsf{Acyc}(Y)$ as a cycle-edge e is deleted. This leads to the construction of the collapse graph of Y and e, which has vertex set the κ-classes of $\mathsf{Acyc}(Y)$. We show that there is a bijection from the set of connected components of this graph to the set of κ-equivalence classes of $\mathsf{Acyc}(Y'_e)$. Moreover, we establish that there is a bijection from the edge set of the collapse graph to the set of κ-equivalence classes of $\mathsf{Acyc}(Y''_e)$. From this and the fact that the collapse graph is a forest the recursion (1) follows. Alternatively, the recursion can be derived through an observation made by Vic Reiner, see [7, Remark 5.5]: the number of equivalence classes of linear orderings under the operations of *(i)* transposition of successive, non-connected generators and *(ii)* cyclic shifts is counted by (1). The bijection between Coxeter elements and acyclic orientations in [11] provides the connection to our setting. Even though the connection of this fact to the enumeration of conjugacy classes of Coxeter elements is straightforward, this does not appear in the literature. Our contribution is an independent and direct proof of this result by examining the acyclic orientations of the Coxeter graph. Additionally, our proof provides insight into the structure of the equivalence classes. We believe that the techniques involved may be useful in extending current results in Coxeter theory, in particular, some from [11].

Let Y be a finite undirected graph with vertex set $\mathrm{v}[Y] = \{1, 2, \ldots, n\}$ and edge set $\mathrm{e}[Y]$. An orientation of Y is represented by a map $O_Y : \mathrm{e}[Y] \longrightarrow \mathrm{v}[Y] \times \mathrm{v}[Y]$, and the graph $G(O_Y)$ is obtained from Y by orienting each edge as given by O_Y. We will use O_Y and $G(O_Y)$ interchangeably when no ambiguity can arise. An orientation O_Y is acyclic if $G(O_Y)$ has no directed cycles. The set of acyclic orientations of Y is denoted $\mathsf{Acyc}(Y)$, and we set $\alpha(Y) = |\mathsf{Acyc}(Y)|$, which can be computed through the well-known recursion relation

$$\alpha(Y) = \alpha(Y'_e) + \alpha(Y''_e)\ . \tag{2}$$

As above, Y'_e and Y''_e are the graphs obtained from Y by deletion and contraction of a fixed edge e, respectively. It is known that there is a bijection between $\mathsf{Acyc}(Y)$ and the set of Coxeter elements of Coxeter group whose Coxeter graph is Y [4, 10]. There is also a bijection between $\mathsf{Acyc}(Y)$ and the set of chambers of the graphic hyperplane arrangement $\mathcal{H}(Y)$ [8].

If v is a source of an acyclic orientation O_Y with degree ≥ 1, then reversing the orientation of all the edges incident to v maps O_Y to a new orientation of Y, which is also acyclic. This is called a *source-to-sink* operation, or a *click*. We define the equivalence relation $\sim_\kappa$ on the set of acyclic orientations for a fixed graph Y by $O_Y \sim_\kappa O'_Y$ if there is a sequence of source-to-sink operations that maps O_Y to O'_Y. Two such orientations are said to be *click-equivalent*, or *κ-equivalent*. We set $\kappa(Y) = |\mathsf{Acyc}(Y)/\sim_\kappa|$.

The set of linear orders on $\mathrm{v}[Y]$ can be represented by the set of permutations of $\mathrm{v}[Y]$, which we denote as S_Y. We write $[\pi]_Y$ for the set of linear orders compatible with the acyclic orientation O_Y induced by π. There is a bijection between $\{[\pi]_Y \mid \pi \in S_Y\}$ and $\mathsf{Acyc}(Y)$, see, e.g. [9]. Let π be the permutation representation of a linear order compatible with O_Y. Note that mapping $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ to $\pi' = (\pi_2, \ldots, \pi_n, \pi_1)$ corresponds to converting π_1 from a source to a sink in O_Y. In general, two distinct acyclic orientations O_Y and O'_Y are κ-equivalent if and only if there exists π compatible with O_Y and π' compatible with O'_Y such that π' can be obtained from π by (i) cyclic shifts and (ii) transpositions of consecutive elements that are not connected in Y. For a given Coxeter group W with generators $S = \{s_i\}_{i=1}^n$ and Coxeter graph Y there is a similar mapping from S_Y into the set of Coxeter elements $C(W)$, and a bijection from $C(W)$ to $\mathsf{Acyc}(Y)$, see [10]. Thus, an acyclic orientation represents a unique Coxeter element, and a source-to-sink operation corresponds to conjugating that element by a particular generator. Therefore, $\kappa(Y)$ is an upper bound for the number of conjugacy classes of Coxeter elements in a Coxeter group whose Coxeter graph is Y, and this bound is known to be sharp in certain cases [11]. A simple induction argument shows that if Y is a tree, then $\kappa(Y) = 1$, and thus all Coxeter elements in a finite Coxeter group are conjugate [4]. In [11], the author shows that if Y contains a single cycle of length n, then $\kappa(Y) = n-1$. This becomes a straightforward corollary of Theorem 1. The recurrence of Theorem 1 appears in several areas of mathematics, and corresponds to the evaluation of the Tutte polynomial at $(1, 0)$, which we describe in Section 4.

2 Preliminary Results

We begin our study of $\kappa(Y)$ by making the following simple observation recorded without proof.

Proposition 1 *Let Y be the disjoint union of undirected graphs Y_1 and Y_2. Then*

$$\kappa(Y) = \kappa(Y_1)\kappa(Y_2) \ . \tag{3}$$

In light of this, we may assume that Y is connected when computing $\kappa(Y)$. An edge of Y that is not contained in any simple cycle of Y is a *bridge*, otherwise it is a *cycle-edge*. The graph obtained from Y by deletion of all bridges is the *cycle graph* of Y and it is denoted $\mathrm{Cycle}(Y)$. Alternatively, an edge e of a connected graph Y is a bridge if the deletion of e disconnects Y. Bridges do not contribute to $\kappa(Y)$ as shown in the following proposition.

Proposition 2 *Let Y be an undirected and graph, and let $e = \{v, w\}$ be a bridge of Y, connecting the disjoint subgraphs Y_1 and Y_2. Then one has the relation*

$$\kappa(Y) = \kappa(Y_1)\kappa(Y_2) \ . \tag{4}$$

Proof Each pair of acyclic orientations $O_{Y_1} \in \mathsf{Acyc}(Y_1)$ and $O_{Y_2} \in \mathsf{Acyc}(Y_2)$ extends to exactly two acyclic orientations of Y by $O_Y = (O_{Y_1}, (v,w), O_{Y_2})$ and $O'_Y = (O_{Y_1}, (w,v), O_{Y_2})$ defined in the obvious way. Clearly, every acyclic orientation of Y is also of one of these forms. Moreover, any click sequence for O'_Y that contains each vertex of Y_2 exactly once and contains no vertices of Y_1 maps O'_Y to O_Y. Hence O_Y and O'_Y are click-equivalent. It follows that $O_Y, O'_Y \in \mathsf{Acyc}(Y)$ are click-equivalent if and only their corresponding acyclic orientations over Y_1 and Y_2 are click-equivalent, and the equality (4) now follows from Proposition 1. □

Proposition 2 gives us the immediate corollary.

Corollary 1 *For any undirected graph Y we have $\kappa(Y) = \kappa(\mathrm{Cycle}(Y))$. In particular, if Y is a forest then $\kappa(Y) = 1$.*

We remark that the first part of this corollary is proven in [11] for the special case where $\mathrm{Cycle}(Y)$ is a circle. The second part is well-known (see, e.g. [4]).

Let $P = (v_1, v_2, \ldots, v_k)$ be a (possibly closed) simple path in Y. The map

$$\nu_P \colon \mathsf{Acyc}(Y) \longrightarrow \mathbb{Z} \tag{5}$$

evaluated at O_Y is the number of edges of the form $\{v_i, v_{i+1}\}$ in Y oriented as (v_i, v_{i+1}) in O_Y (positive edges) minus the number of edges oriented as (v_{i+1}, v_i) in O_Y (negative edges).

Lemma 1 *Let P be a simple closed path in the undirected graph Y. The map ν_P extends to a map $\nu_P^* \colon \mathsf{Acyc}(Y)/\!\sim_\kappa \longrightarrow \mathbb{Z}$.*

Proof Let $c(O_Y) = O'_Y$ where $c = c_v$ is a click of a single vertex v. If v is not an element of P then clearly $\nu_P(O_Y) = \nu_P(O'_Y)$. On the other hand, if v is contained in P then c maps one positive edge into a negative edge and vice versa. The general case follows by induction on the length of the click sequence. □

Lemma 1 will be used extensively in the proof of the main result in the next section.

3 Proof of the Main Theorem

From Proposition 2 it is clear that for the computation of $\kappa(Y)$ all bridges may be omitted. We now turn our attention to the role played by cycle-edges in determining $\kappa(Y)$ and to the proof of the recursion relation

$$\kappa(Y) = \kappa(Y'_e) + \kappa(Y''_e) \tag{6}$$

of Theorem 1 valid for any cycle-edge e of Y. We set $e = \{v, w\}$ in the following.

First, define $\iota_1 \colon \mathsf{Acyc}(Y''_e) \longrightarrow \mathsf{Acyc}(Y)$ as the map that sends $O_{Y''} \in \mathsf{Acyc}(Y''_e)$ to $O_Y \in \mathsf{Acyc}(Y)$ for which $O_Y(e) = (v, w)$ and for which all other edge orientations are inherited. The map $\iota_2 \colon \mathsf{Acyc}(Y''_e) \longrightarrow \mathsf{Acyc}(Y)$ is defined analogously, but orients e as (w, v).

Proposition 3 *The maps* $\iota_{1,2}\colon \mathsf{Acyc}(Y_e'') \longrightarrow \mathsf{Acyc}(Y)$ *extend to well-defined maps*

$$\iota_{1,2}^*\colon \mathsf{Acyc}(Y_e'')/\sim_\kappa \longrightarrow \mathsf{Acyc}(Y)/\sim_\kappa \ . \tag{7}$$

Proof For $\phi \in \{\iota_1, \iota_2\}$ and for any click-sequence c of $O_{Y''} \in \mathsf{Acyc}(Y'')$ we have the commutative diagram

$$\begin{array}{ccc} O_{Y''} & \xrightarrow{\ c\ } & O'_{Y''} \\ \phi\downarrow & & \downarrow\phi \\ O_Y & \xrightarrow{\ c'\ } & O'_Y \end{array} \tag{8}$$

where the click-sequence c' over Y is constructed from the click-sequence c over Y'' by insertion of w after (resp. before) every occurrence of v in c for ι_1 (resp. ι_2). $\square$

Proposition 4 *Let e be a cycle-edge. For any $[O_{Y''}] \in \mathsf{Acyc}(Y'')$ we have $\iota_1^*([O_{Y''}]) \neq \iota_2^*([O_{Y''}])$.*

Proof Let P be any simple closed path containing e and oriented so as to include (v, w). From the definition of ι_1 and ι_2 we conclude that $\nu_P\big(\iota_1(O_{Y''})\big) = \nu_P\big(\iota_2(O_{Y''})\big) + 2$, and the proposition follows by Lemma 1.

Proposition 5 *The maps $\iota_{1,2}^*$ are injections.*

Proof We prove the statement for ι_1^*. The proof for ι_2^* is analogous. Assume $[O_{Y''}] \not\sim_\kappa [O'_{Y''}]$ both map to $[O_Y]$ under ι_1^*. By construction, any elements $O_{Y''}$ and $O'_{Y''}$ of the respective κ-classes have ι_1-images with e oriented as (v, w). Moreover, for any image point of ι_1 there is no directed path from v to w of length ≥ 2, and there is no directed path from w to v. We may also assume that v is a source in both $O_{Y''}$ and $O'_{Y''}$. From this it is clear that v and w belong to successive layers in the acyclic orientations O_Y and O'_Y.

Let c be a click-sequence taking $\iota_1(O_{Y''})$ to $\iota_1(O'_{Y''})$. Again by construction, we may assume that any occurrence of v in c is immediately followed by w. This follows since v and w have to occur equally many times in c, and from the fact that v and w belong to successive layers in O_Y and O'_Y. If v and w were not consecutive in c it could only be because c is of the form $c = \ldots v\, v_1 \ldots v_k\, w_1 \ldots w_r\, w \ldots$ where the v_i's belong to the same layer as v and the w_i's belong to the same layer as w. A layer is in particular an independent set, and it is clear that the sequence $c' = \ldots v_1 \ldots v_k\, v\, w\, w_1 \ldots w_r \ldots$ obtained from c by changing the order of v and w also maps O_Y to O'_Y.

The click-sequence c'' obtained from c' by deleting every occurrence of w is a click-sequence mapping $O_{Y''}$ to $O'_{Y''}$, which contradicts the assumption that $[O_{Y''}] \not\sim_\kappa [O_{Y''}]'$. $\square$

Consequently, any κ-class $[O_Y]$ contains at most one set of the form $\iota_1([O_{Y''}])$, and at most one set of the form $\iota_2([O_{Y''}])$.

Proposition 6 *Let e be a cycle-edge of the undirected graph Y. For each pair of distinct κ-classes $[O_Y]$ and $[O_Y]'$ there is at most one κ-class $[O_{Y''}]$ such that $\{\iota_1^*([O_{Y''}]), \iota_2^*([O_{Y''}])\} = \{[O_Y], [O_Y]'\}$.*

Proof Assume this is not not the case, and that there in fact is another class $[O_{Y''}]'$ with the same property. Since both maps $\iota_{1,2}^*$ are injective it then follows (up to relabeling) that $\iota_1^*([O_{Y''}]) = \iota_2^*([O_{Y''}]') = [O_Y]$ and $\iota_1^*([O_{Y''}]') = \iota_2^*([O_{Y''}]) = [O_Y]'$. By the same argument as in the proof of Proposition 4 it follows using $[O_{Y''}]$ that $\nu_P^*([O_Y]) = \nu_P^*([O_Y]') + 2$. On the other hand, by using $[O_{Y''}]'$ if follows that that $\nu_P^*([O_Y]') = \nu_P^*([O_Y]) + 2$, which is impossible. □

Definition 1 *Let e be a cycle-edge of the undirected graph Y. The collapse graph $\mathfrak{C}_e(Y)$ of Y and e is the graph with vertex set $\mathsf{Acyc}(Y)/{\sim_\kappa}$ and edge set $\big\{\{\iota_1^*([O_{Y''}]), \iota_2^*([O_{Y''}])\} \,\big|\, [O_{Y''}] \in \mathsf{Acyc}(Y'')/{\sim_\kappa}\big\}$.*

Note that by Proposition 6, the graph $\mathfrak{C}_e(Y)$ is simple, and by Proposition 4, it has no singleton edges (i.e. self-loops).

A *line graph* on k vertices has vertex set $\{1, 2, \ldots, k\}$ and edges $\{i, i+1\}$ for $1 \leq i \leq k-1$.

Proposition 7 *Let e be a cycle-edge of the undirected graph Y. The collapse graph $\mathfrak{C}_e(Y)$ is isomorphic to a disjoint collection of line graphs.*

Proof By Definition 1 and the remark following it, each vertex in the collapse graph has degree ≤ 2. Thus it is sufficient to show that $\mathfrak{C}_e(Y)$ contains no cycles. By the now familiar argument using ν_P for some path containing e, two adjacent κ-classes in $\mathfrak{C}_e(Y)$ differ in their ν^*-values by precisely 2. Assume $\mathfrak{C}_e(Y)$ contains the subgraph (up to relabeling)

$$\underset{\iota_1^*(A_1)}{A''} \longrightarrow \underset{\iota_2^*(A_1) = \iota_1^*(A_2)}{A} \longrightarrow \underset{\iota_2^*(A_2)}{A'} \quad , \tag{9}$$

for unique κ-classes $A_{1,2} \in \mathsf{Acyc}(Y'')/{\sim_\kappa}$. Clearly $\nu_P^*(A'') = \nu_P^*(A) + 2$ and $\nu_P^*(A') = \nu_P^*(A) - 2$. We now have the following situation: (i) the ν^*-values of adjacent vertices in $\mathfrak{C}_e(Y)$ differ by precisely 2, and (ii) the value of $\nu*$ increases by 2 across each edge in the A''-direction relative to A and decreases by 2 across each edge in the A'-direction relative to A. If the subgraph in (9) was a part of cycle of length ≥ 3 in $\mathfrak{C}_e(Y)$, then by (ii) there must be some pair of adjacent vertices for which ν^* differs by at least 4. But this is impossible by (i), hence $\mathfrak{C}_e(Y)$ contains no cycles and the proof is complete. □

Proposition 8 *Let e be a cycle-edge of the undirected graph Y. Then κ-classes on the same connected component in $\mathfrak{C}_e(Y)$ are contained in the same κ-class in $\mathsf{Acyc}(Y')/{\sim_\kappa}$.*

Proof It is sufficient to show this for adjacent vertices in $\mathfrak{C}_e(Y)$ – the general result follows by induction. Clearly, $O_Y {\sim_\kappa} O'_Y$ implies $O_{Y'} {\sim_\kappa} O'_{Y'}$. Adjacent vertices in $\mathfrak{C}_e(Y)$ contain elements that only differ in their orientations of e and hence become κ-equivalent upon deletion of e. The proof follows. □

Proposition 9 *There is a bijection between the connected components in* $\mathfrak{C}_e(Y)$ *and* $\mathsf{Acyc}(Y'_e)/\sim_\kappa$.

Proof Let n_c denote the number of connected components of $\mathfrak{C}_e(Y)$. By Proposition 8 if $[O_Y]$ and $[O'_Y]$ are connected in $\mathfrak{C}_e(Y)$ then both classes are contained in the same κ-class over Y', and thus $n_c \leq \kappa(Y')$.

It is clear that a κ-class contains all acyclic orientations for which there are representative permutations that are related by a sequence of adjacent transpositions of non-connected vertices in Y and cyclic shifts. Upon deletion of the cycle-edge e the adjacent transposition of the endpoints of e becomes permissible, and thus two distinct κ-classes in Y containing acyclic orientations that only differ on e are contained within the same κ-class over Y'. By reference to the underlying permutations, it follows that two κ-classes in Y are contained within the same κ-class in Y' if and only if there is a sequence of κ-classes in Y where consecutive elements in the sequence contain acyclic orientations that differ precisely on e. By the definition of $\mathfrak{C}_e(Y)$ it follows that all κ-classes over Y that merge to be contained within one κ-class in Y' upon deletion of e are contained within the same connected component of $\mathfrak{C}_e(Y)$, and thus $n_c \geq \kappa(Y')$. □

Proof (Theorem 1) Upon deletion of e in Y two or more κ-classes over Y may merge to be contained with the same κ-class over Y'. By Proposition 9 the number of κ-classes over Y' equals the number of connected components in $\mathfrak{C}_e(Y)$. If a connected component in $\mathfrak{C}_e(Y)$ contains m distinct κ-classes of Y then by Proposition 6 there are $m-1$ unique corresponding κ-classes over Y''. Thus for each component of $\mathfrak{C}_e(Y)$ we have a relation precisely of the form (1) for the κ-classes involved. The theorem now follows. □

4 Related Enumeration Problems

In this section we relate the problem of computing $\kappa(Y) = |\mathsf{Acyc}(Y)/\sim_\kappa|$ to two other enumeration problems where the same recurrence holds. We will show how these problems are equivalent, and additionally, how they all can be computed through an evaluation of the Tutte polynomial. As a corollary we obtain a transversal of $\mathsf{Acyc}(Y)/\sim_\kappa$.

In [2] the notion of *cut-equivalence* of acyclic orientations is studied. Recall that a *cut* of a graph Y is a partition of the vertex set into two classes $\mathrm{v}[Y] = V_1 \sqcup V_2$, and where $[V_1, V_2]$ is the set of (cut-)edges between V_1 and V_2. A cut of a directed graph of the form $G(O_Y)$, which we simply refer to as a cut of O_Y, is *oriented* with respect to O_Y if the edges of $[V_1, V_2]$ are all directed from V_1 to V_2, or are all directed from V_2 to V_1.

Definition 2 *Two acyclic orientations* O_Y *and* O'_Y *are* cut-equivalent *if the set* $\{e \in \mathrm{v}[Y] \mid O_Y(e) \neq O'_Y(e)\}$ *is (i) empty or is (ii) an oriented cut with respect to either* O_Y *or* O'_Y.

The study of cut-equivalence in [2] was done outside the setting of Coxeter theory, and here we provide the connection.

Proposition 10 *Two acyclic orientations of Y are κ-equivalent if and only if they are cut-equivalent.*

Proof Suppose distinct elements O_Y and O'_Y in $\mathsf{Acyc}(Y)$ are cut-equivalent, and without loss of generality, that all edges of $[V_1, V_2]$ are oriented from V_1 to V_2 in O_Y. A click-sequence containing each vertex of V_1 precisely once maps O_Y to O'_Y, thus $O_Y \sim_\kappa O'_Y$.

Conversely, suppose that $O_Y \sim_\kappa O'_Y$, where O'_Y is obtained from O_Y by a click-sequence containing a single vertex v. Then O_Y and O'_Y are cut-equivalent, with the partition being $\{v\} \sqcup \mathrm{v}[Y] \setminus \{v\}$. □

Obviously, the recurrence relation in (1) holds for the enumeration of both cut-equivalence and κ-equivalence classes.

Definition 3 *The* Tutte polynomial *of an undirected graph Y is defined as follows. If Y has b bridges, ℓ loops, and no cycle-edges, then $T(Y, x, y) = x^b y^\ell$. If e is a cycle-edge of Y, then*

$$T(Y, x, y) = T(Y'_e, x, y) + T(Y''_e, x, y) \; .$$

We remark that it is well-known that the number of acyclic orientations of a graph Y can be evaluated as $\alpha(Y) = T(Y, 2, 0)$. It was shown in [2] that the number of cut-equivalence classes can be computed through an evaluation of the Tutte polynomial as $T(Y, 1, 0)$, and thus $\kappa(Y) = T(Y, 1, 0)$.

It is known that $T(Y, 1, 0)$ counts several other quantities, some of which can be found in [3]. One of these is $|\mathsf{Acyc}_v(Y)|$, the number of acyclic orientations of Y where a fixed vertex v is the unique source. As the next result shows, there is a bijection between $\mathsf{Acyc}(Y)/\sim_\kappa$ and $\mathsf{Acyc}_v(Y)$.

Proposition 11 *Let Y be a connected graph Y. For any fixed $v \in \mathrm{v}[Y]$ there is a bijection*

$$\phi_v \colon \mathsf{Acyc}(Y)/\sim_\kappa \longrightarrow \mathsf{Acyc}_v(Y) \; .$$

Proof Since $\kappa(Y) = |\mathsf{Acyc}(Y)/\sim_\kappa| = T(Y, 1, 0) = |\mathsf{Acyc}_v(Y)|$ it is sufficient to show that that there is a surjection $\beta_v \colon \mathsf{Acyc}(Y)/\sim_\kappa \longrightarrow \mathsf{Acyc}_v(Y)$. Let $A \in \mathsf{Acyc}(Y)/\sim_\kappa$, let $O_Y \in A$, and let c be a maximal length click-sequence not containing the vertex v. This click-sequence is finite since Y is connected. The orientation $c(O_Y)$ has v as a unique source, since otherwise c would not be maximal, and the proof is complete. □

From Proposition 11 we immediately obtain:

Corollary 2 *For any vertex v of Y the set $\mathsf{Acyc}_v(Y)$ is a transversal of $\mathsf{Acyc}(Y)/\sim_\kappa$.*

In light of the results in this section, the recurrence in (1) may also be proven by showing that the map β_v is injective. However, our proof offers insight into the structure of the κ-classes, and it is our hope that this may lead to new techniques for studying conjugacy classes of Coxeter groups.

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