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What Moser *Could* Have Asked: Counting Hamilton Cycles in Tournaments

Neil J. Calkin, Beth Novick and Hayato Ushijima-Mwesigwa

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Abstract

Moser asked for a construction of explicit tournaments on n vertices having at least $(\frac{n}{3e})^n$ Hamilton cycles. We show that he could have asked for rather more.

1 Introduction

... the cycle has taken us up through forests.

Robert M. Pirsig

In his classic book on tournaments, Moon [4, Section 10] discusses the question of exhibiting tournaments with a large number of Hamilton cycles. He poses the question (Exercise 4, attributed to Moser), of constructing a tournament on n vertices having at least $(\frac{n}{3e})^n$ Hamilton cycles. Presumably, the intended construction is to take three tournaments, T_1, T_2, T_3 , on $\frac{n}{3}$ vertices, and construct a new tournament $C_3(T_1, T_2, T_3)$ by orienting all edges from T_1 to T_2 , T_2 to T_3 , and T_3 to T_1 (See Figure 1). The number of Hamilton cycles

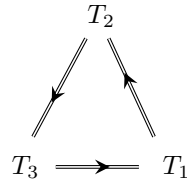


Figure 1: $C_3(T_1, T_2, T_3)$

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in $C_3(T_1, T_2, T_3)$ which do not use any edges internal to T_1, T_2 , or T_3 is

$$\frac{\left(\frac{n}{3}\right)!^3}{\frac{n}{3}} \sim \sqrt{\frac{8\pi^3 n}{3}} \left(\frac{n}{3e}\right)^n > \left(\frac{n}{3e}\right)^n.$$

In this note, we show that this construction has many more Hamilton cycles. Indeed, if T_1, T_2 , and T_3 are all transitive, we show that the number of Hamilton cycles is asymptotic to $\frac{1}{(1-\log 2)} \frac{(n-1)!}{(3 \log 2)^n}$.

2 Background and Definitions

A *tournament* is an oriented, complete graph. A *Hamilton cycle* or *path* in a tournament T , is a spanning directed cycle or directed path in T . A tournament with no directed cycles is called *transitive*.

Counting Hamilton paths and cycles in tournaments is a very old problem, dating back to the 1940's: in one of the first applications of the probabilistic method, Szele [6] showed that the expected number of Hamilton paths in a random tournament is $\frac{n!}{2^{n-1}}$, therefore showing that there exists a tournament on n vertices with at least this many Hamilton paths. The same argument shows that there exists a tournament with at least $\frac{(n-1)!}{2^n}$ Hamilton cycles. Moon observed that it seems difficult to give explicit tournaments with at least this many Hamilton cycles.

Deep results of Cuckler [3] show that every regular tournament on n vertices has at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles.

Given tournaments T_1, T_2, T_3 , we can construct a tournament $C_3(T_1, T_2, T_3)$ by orienting all edges from T_1 to T_2 , T_2 to T_3 , and T_3 to T_1 . We will call such tournaments *triangular*. Wormald [8] showed that if T_1, T_2, T_3 are random tournaments, then the expected number of Hamilton cycles is $2 \frac{(n-1)!}{2^n}$.

We show that *all* triangular tournaments have a relatively large number of Hamilton cycles, even in the extreme case when they constructed from transitive tournaments.

Let $S(m, k)$ denote the Stirling number of the second kind, that is, $S(m, k)$ is the number of set partitions of $\{1, 2, \dots, m\}$ into exactly k parts.

3 Main Result

Theorem 1. *Let T_1, T_2 , and T_3 be any tournaments on m_1, m_2 , and m_3 vertices respectively. Then the number H of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is at least*

$$H \geq \sum_{k=1}^{\min\{m_1, m_2, m_3\}} S(m_1, k) S(m_2, k) S(m_3, k) \frac{k!^3}{k}, \quad (1)$$

with equality when T_1, T_2 , and T_3 are transitive.

Corollary 2. If T_1, T_2 , and T_3 are transitive tournaments on $\frac{n}{3}$ vertices, the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is asymptotic to

$$\frac{1}{(1 - \log 2)} \frac{(n-1)!}{(3 \log 2)^n} \simeq 3.25889 \frac{(n-1)!}{(2.07944)^n}. \quad (2)$$

Proof of Theorem 1. Take any Hamilton cycle C , of $C_3(T_1, T_2, T_3)$, and consider C restricted to T_1, T_2 , and T_3 . Since a Hamilton cycle meets every vertex in T_1, T_2 , and T_3 exactly once, C visits each subtournament the same number of times, say k . Hence for each T_i , C will induce a collection of k disjoint paths that cover the vertices of T_i . We will refer to such a collection of k paths as a k -path cover. Similarly, given k -path covers for T_1, T_2, T_3 , we can construct a Hamilton cycle by joining these k -path covers together. The number of ways of doing this is $k!^3/k$. Thus, if $P(T_i, k)$ denotes the number of k -path covers of T_i , then the number of Hamilton cycles of $C_3(T_1, T_2, T_3)$ which induce k -path covers in T_1, T_2 , and T_3 is

$$P(T_1, k)P(T_2, k)P(T_3, k) \frac{k!^3}{k}. \quad (3)$$

It follows that the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is

$$\sum_{k=1}^{\min\{m_1, m_2, m_3\}} P(T_1, k)P(T_2, k)P(T_3, k) \frac{k!^3}{k}.$$

For any set partition of the vertex set of T_i into k nonempty sets, each part will induce a subtournament of T_i . Rédei [5] showed that every tournament has a Hamilton path, thus each partition into k sets will induce at least one k -path cover of T_i . Therefore the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is at least

$$\sum_{k=1}^{\min\{m_1, m_2, m_3\}} S(m_1, k)S(m_2, k)S(m_3, k) \frac{k!^3}{k}$$

as claimed.

In the case that each T_i is transitive, each subtournament will have exactly one Hamilton path, hence we have equality in (1). \square

Proof of Corollary 2. Suppose now that each T_i is a transitive tournament on m vertices, then the number of Hamilton cycles in $C_3(T_1, T_2, T_3)$ is equal to

$$\sum_{k=1}^m S(m, k)^3 \frac{k!^3}{k}. \quad (4)$$

As with many combinatorial sums, the summands in (4) are approximated rather well by a normal distribution. Indeed, if we let

$$\mu = \frac{1}{2 \log 2} \quad \text{and} \quad \sigma = \frac{\sqrt{1 - \log 2}}{2 \log 2},$$

define $f(m) = \sum_{k=1}^m S(m, k)k!$, and write $p(m, k) = \frac{S(m, k)k!}{f(m)}$, then Bender [2] shows that $p(m, k)$ is asymptotically normal with mean μm and variance $\sigma^2 m$. Hence, $p(m, k)^3$ is also proportional to a normal distribution, at least in a range of k close to μm . This allows us to approximate the sum $\sum_{k=1}^m S(m, k)^3 \frac{k!^3}{k}$ by an integral, showing that

$$\sum_{k=1}^m S(m, k)^3 \frac{k!^3}{k} \sim f(m)^3 \frac{3^{\frac{1}{2}} 2^{\frac{1}{2}} \pi^{\frac{1}{2}}}{3\mu\sigma^2 (2\pi)^{\frac{3}{2}} m^2}.$$

From Wilf [7, p. 176], we know that

$$f(m) \sim \frac{m!}{2(\log 2)^{m+1}},$$

Therefore with $n = 3m$, two applications of Stirling's approximation for $n!$ yields

$$\begin{aligned} \sum_{k=1}^m S(m, k)^3 \frac{k!^3}{k} &\sim \left(\frac{m!}{2(\log 2)^{m+1}} \right)^3 \frac{3^{\frac{1}{2}} 2^{\frac{1}{2}} \pi^{\frac{1}{2}}}{3\mu\sigma^2 (2\pi)^{\frac{3}{2}} m^2} \\ &\sim \frac{\sqrt{2\pi n}}{n(1 - \log 2)} \left(\frac{n}{3e \log 2} \right)^n \\ &\sim \frac{1}{(1 - \log 2)} \frac{(n-1)!}{(3 \log 2)^n} \end{aligned}$$

□

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