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A Carleman type estimate for the Mindlin-Timoshenko plate model

Jason A. Kurz
Clemson University, jkurz@clemson.edu

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Jason A. Kurz
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Abstract

This thesis focuses on results concerning providing a Carleman type estimate for the Mindlin-Timoshenko plate equations. The main approach is to provide an estimate for each of the three equations in the model then present these estimates in totality as a singular Carleman estimate for the entire model. The initial equation in the model is a simple two dimensional hyperbolic partial differential equation known as the wave equation. Prior research has been done for this type of equation and will be applied to provide the Carleman estimate for the first equation in the model. The estimate for the second and third equations will be derived by first establishing a point-wise inequality for the principal part of the equation multiplied by an exponential weight. After establishing a suitable pseudo-convex function for the exponential weight factor, specifications will be applied to the established point-wise estimates which will lead to the Carleman type estimates and their corresponding integral inequalities.

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Chapter 1

Introduction

1.1 Background

The motivations behind the study of the model explored in the thesis has developed from the classical Euler-Bernoulli beam theory as well as Kirchoff-Love plate models [2, 10]. In more recent years, models which account for shear deformations have been of more interest and the models in classical beam theory are limited in their description of plates or beams experiencing high-frequency vibrations [2, 10]. A model accounting for the transverse shear deformation occurring to the plate involving two shear angles was considered by Reissner [2, 11]. Reissner's model possessed several deviations from classical plate theory, including allowing for a change in the thickness of the plate due to stresses [11]. These were also changes from Timoshenko's earlier proposed model, which considered the displacement of a beam taking into account a single shear angle of its filament [2, 10, 15]. A later model was proposed by Mindlin, independent of Reissner, that also considered two shear angles, and has been foundational in the development of modern plate theory [2, 9, 10]. The Mindlin-Timoshenko model considered for this present paper was considered in Lagnese [4] with explorations of the systems stability and well-posedness researched by Pei et al. [10], Jorge Silva et al. [13], Grobbelaar-Van Dalsen [2], and Fernandez Sare [12]. The Mindlin-Timoshenko model is the one of interest for the purposes of the research presented in this paper with the goal of presenting a Carleman type estimate for the model.

1.2 The Model

This paper will be studying the Mindlin-Timoshenko plate model in the two dimensional case given by

$$\begin{cases} \rho h w_{tt} - K\Delta w - K(\phi_x + \psi_y) = 0, & \text{in } \Omega \times [0,T] \\ \frac{\rho h^3}{12}\psi_{tt} - D\left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}\right) - D\left(\frac{1+\mu}{2}\phi_{xy}\right) + K(\psi + w_x) = 0, & \text{in } \Omega \times [0,T] \\ \frac{\rho h^3}{12}\phi_{tt} - D\left(\phi_{yy} + \frac{1-\mu}{2}\phi_{xx}\right) - D\left(\frac{1+\mu}{2}\psi_{xy}\right) + K(\phi + w_y) = 0 & \text{in } \Omega \times [0,T] \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} (w(x,y,0), \psi(x,y,0), \phi(x,y,0)) = (w_0, \psi_0, \psi_0) \in (H_0^1(\Omega))^3 \\ (w_t(x,y,0), \psi_t(x,y,0), \phi_t(x,y,0)) = (w_1, \psi_1, \psi_1) \in (L^2(\Omega))^3 \end{cases}$$

and boundary conditions

$$w = \psi = \phi = 0 \quad \text{on } \partial\Omega \times [0,T]$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain and ρ, h, D and K are positive constants representing the mass per unit surface area, thickness of the plate, flexural rigidity and shear modulus respectively [12, 10, 13]. Notice w is displacement of the plate from the central plane in the normal direction to the mid-surface plane, while ϕ and ψ are the angles of shear deformation [12]. The constant μ is referred to as Poisson's ratio constrained by $0 < \mu < \frac{1}{2}$ in physical situations [12, 10, 13]. The term $\frac{1-\mu}{2}$ will play a central role in parts of this paper and will be denoted as a for the purposes of easing the notation. For the purposes of the results presented, the simplified model given by

$$w_{tt} - \Delta w - (\phi_x + \psi_y) = 0, \quad \text{in } \Omega \times [0,T] \tag{1.2}$$

$$\psi_{tt} - \left(\psi_{xx} + \frac{1-\mu}{2}\psi_{yy}\right) - \frac{1+\mu}{2}\phi_{xy} + (\psi + w_x) = 0, \quad \text{in } \Omega \times [0,T] \tag{1.3}$$

$$\phi_{tt} - \left(\frac{1-\mu}{2}\phi_{xx} + \phi_{yy}\right) - \frac{1+\mu}{2}\psi_{xy} + (\phi + w_y) = 0, \quad \text{in } \Omega \times [0,T] \tag{1.4}$$

will be used for this paper.

1.3 Carleman Estimate for Wave Equation in 2D

The goal of the present paper is to show a Carleman type estimate for the aforementioned Mindlin-Timoshenko plate model with constants normalized. While, care must be taken in handling the term $\frac{1-\mu}{2}$ in equations (1.3) and (1.4), equation (1.2) is simply the wave equation in two dimensions. As such, the work of Lasiecka, Triggiani, and Zhang [6] can be directly applied to (1.2) to provide an estimate. Following Lasiecka et al.'s [6] results, consider the pseudo-convex function $\tilde{P}$ in $\Omega \times [0, T]$ defined by

$$\tilde{P}(x, y, t) \equiv (x - x_0)^2 + (y - y_0)^2 - c\left(t - \frac{T}{2}\right)^2$$

where $(x_0, y_0) \notin \bar{\Omega}$ and the constants T and c are such that $0 < T$ and $0 < c < 1$. Then, defining $E_w(t)$ as follows:

$$E_w(t) \equiv \frac{1}{2}\int_\Omega w_t^2 + |\nabla w|^2 dxdy$$

and choosing the constants $c, \tilde{\sigma}, \delta$ and ρ as specified by Lasiecka, Triggiani and Zhang [6] (see Chapter 3 also) we have, for $\tau > 0$ sufficiently large and arbitrary $\epsilon > 0$ small, the one parameter family of estimates

$$\begin{aligned} &BT|_\Sigma^w + 2\int_0^T \int_\Omega e^{2\tau\tilde{P}} f^2 dxdydt + C_{1,T} e^{2\tau\tilde{\sigma}} \int_{[Q(\tilde{\sigma})]^c} w^2 dxdydt \\ &\geq (\epsilon\tau\rho - 2C)\int_0^T \int_\Omega e^{2\tau\tilde{P}}(w_t^2 + |\nabla w|^2) dxdydt \\ &+ (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\tilde{\sigma})} e^{2\tau\tilde{P}} w^2 dxdydt - C_T\tau^3 e^{-2\tau\delta}[E_w(0) + E_w(T)] \end{aligned} \tag{1.5}$$

where $f \equiv \psi_x + \phi_y$, the region $Q(\tilde{\sigma})$ is given by

$$Q(\tilde{\sigma}) \equiv \left\{(x, y, t) : (x, y) \in \Omega, 0 \leq t \leq T, \tilde{P}(x, y, t) \geq \sigma > 0\right\},$$

and the boundary terms are

$$
\begin{aligned}
BT|_{\Sigma}^{w} &= 2\tau \int_0^T \int_{\partial\Omega} e^{2\tau\tilde{P}}[w_t^2 - |\nabla w|^2]\nabla d \cdot \nu dSdt \\
&+ 8c\tau \int_0^T \int_{\partial\Omega} e^{2\tau\tilde{P}} \left(t - \frac{T}{2}\right) w_t \frac{\partial w}{\partial \nu} dSdt + 8\tau \int_0^T \int_{\partial\Omega} e^{2\tau\tilde{P}} [\nabla d \cdot \nabla w] \frac{\partial w}{\partial \nu} dSdt \\
&+ 4\tau^2 \int_0^T \int_{\partial\Omega} e^{2\tau\tilde{P}} \left[4(x-x_0)^2 + 4(y-y_0)^2 - 4c^2\left(t - \frac{T}{2}\right)^2 + \frac{\alpha\tau}{2}\right] w \frac{\partial w}{\partial \nu} dSdt \\
&+ 2\tau \int_0^T \int_{\partial\Omega} e^{2\tau\tilde{P}} \left\{8\tau^2\left[(x-x_0)^2 + (y-y_0)^2 - c^2\left(t - \frac{T}{2}\right)^2\right] + \tau(\alpha - 2c - 4)\right\} w^2 \nabla d \cdot \nu dSdt.
\end{aligned}
$$

with d defined as

$$d(x,y) = (x-x_0)^2 + (y-y_0)^2$$

for $(x,y) \in \Omega$ and a fixed point $(x_0, y_0) \notin \Omega$. The constant α in the boundary terms is defined as

$$\alpha \equiv 3 - 2c + k \text{ for } 0 < k < 1$$

such that we have

$$4 - 2c - \alpha \geq \rho \equiv \alpha - 2c > 0.$$

For the other two equations careful consideration must be taken to observe how the constant $\frac{1-\mu}{2}$ impacts this process. This constant acts as a weight on one of the terms of the two dimensional Laplacian in each of (1.3) and (1.4) preventing them from following the true wave equation model. The final section of the paper will then establish a sum of the estimates for an overarching estimate of the entire model.

1.4 Some Literature on Carleman estimates

The origination of the use of exponential weights can be traced to a mathematician named Carleman [1] in 1939. Carleman's intent was to apply these estimates to prove uniqueness in the what is known as the Cauchy Problem in two variables. It was the mathematician Hörmander who realized the implications of this notion of Carleman's, which would lead to becoming a mainstay for all related work in the field [3, p.61]. Hörmander continued to popularize Carleman's approach and perfected the concept to a more broad class of differential operators. The general representation for this Carleman estimate is given by

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int |D^{\alpha} u e^{2\tau\varphi}|^2 dx \leq C \int |P u e^{2\tau\varphi}|^2 dx.$$

Hörmander subsequently used this estimate to prove what is known as the *Unique Continuation Property* defined below

Definition 1.4.1 (Unique Continuation Property)**.** Consider the PDE $P(x, D)u = 0$. Then suppose u to be the solution on some bounded domain $\Omega \subset \mathbb{R}^n$ and $u = 0$ for some $\varphi(x) > 0$, where the function $\varphi : \Omega \to \mathbb{R}$ defines a smooth hypersurface in the domain Ω, meaning φ is smooth and $\nabla\varphi \neq 0$ on $\varphi = 0$. This property would then imply $u = 0$ on a neighborhood of $\varphi = 0$. [8]

These early results were only applicable, however, when involving solutions which assumed to be compactly supported. Thus, these Carleman estimates did not contain boundary terms which play a vital role in boundary control problems [8]. To emphasize this deficiency in the estimates lacking boundary terms, homogenizing the Cauchy data (a known simple process) produces a term in the right-hand side of the estimate involving norms of boundary traces that are a half derivative higher than the norm of u on the left-hand side of the estimate [8]. This stresses the need for the addition of boundary terms to the classical Carleman estimate since they are deficient in providing decent results when applied to boundary value problems [8]. This issue was addressed by two different approaches that were developed independently.

The development of improved Carleman type inequalities, which provided good results for solutions of boundary value problems can be attributed to two originating sources that addressed the issue rather differently. The first source is the mathematician D. Tataru [14] at the University of Virginia while the second is Lavrentev–Romanov–Shishataskii [7] of the Novosibirski school. These papers established to camps of though for how to produce boundary terms in the estimates [8]. The idea behind Tataru's work was motivated by extending the main Carleman estimate to general psuedo–differential operators [8]. This results in certain structures that need to satisfy geometrical properties, including a surface which must be psuedo–convex. Tataru's work was developed from the work of of Lasiecka-Triggiani [5] which developed a sharp Carleman type estimate specifically for second-order hyperbolic equations such as a wave. These estimates, were obtained using a type of differential multiplier, which differed depending on the exact partial differential equation to which it was applied [8]. In contrast, Lavrentev–Romanov–Shishaskii [7] approached the problem of producing boundary terms in the estimate via a format which was much more computationally focused. Their method was to establish an initial point–wise Carleman estimate with the resulting integral form of this estimate [8]. This was the inspiration behind the subsequent work of Lasiecka-Triggiani-Zhang [6], the primary source for the work in this paper. In their paper, Lasiecka–Triggiani–Zhang [6] worked via the method produced in the Lavrentev camp by establishing a fundamental initial point–wise inequality for the general second order hyperbolic equation that was used to produce a a one parameter family of point–wise Carleman estimates [6].

The Carleman estimates derived for the Mindlin-Timoshenko equations follow the process established by Lasiecka–Triggiani–Zhang wherein we establish an initial point-wise estimate for the second and third equations in the system. This estimate will then, via careful selection of an appropriate pseudo–convex function and other specifications, will ultimately yield point–wise Carleman estimates, and followed by the corresponding integral inequalities. The final estimate is expressed in terms of these point–wise integral inequalities.

The organization for the main result of this thesis is as follows: In Chapter 2 we prove the main point–wise estimates for the equations of two shear angles in the model. In Chapter 3 we will derive the Carleman estimates by introducing a suitable pseudo-convex function that is needed for the special structure of the equations (1.3) and (1.4).

Chapter 2

A Fundamental Lemma

The following Lemma is the main result which consists of pointwise estimates for the equations (1.3), (1.4) in the Mindlin-Timoshenko model discussed in the introduction. The proof for the first inequality is shown while the second is simply stated due to its derivation following a similar process as the proof of Lemma 1.

2.1 Lemma

Let

$$\psi(x,y,t) \in C^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t);\ \ell(x,y,t) \in C^3(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t);\ \zeta(x,y,t) \in C^2 \text{ in } t \text{ and } C^1 \text{ in } x,y$$

be given and set $\theta(x,y,t) = e^{\ell(x,y,t)}$. Additionally, set

$$v(x,y,t) = \theta(x,y,t)\psi(x,y,t)$$
$$A = (\ell_t^2 - \ell_{tt}) - (\ell_x^2 - \ell_{xx}) - a(\ell_y^2 - \ell_{yy}).$$

Then, letting $\epsilon > 0$ be arbitrary, we have the following pointwise inequality (for ease of computation, we use the substitution $a = \frac{1-\mu}{2}$ throughout the statement and proof of the lemma)

$$
\begin{aligned}
&\theta^2[\psi_{tt} - (\psi_{xx} + a\psi_{yy})]^2 - \frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(\psi_t^2 + \psi_x^2 + a\psi_y^2) + 4\psi_t(\ell_x\psi_x + a\ell_y\psi_y) + 2(2\ell_x^2 + 2a\ell_y^2 - 2\ell_t^2 + \zeta)\psi\psi_t\right.\\
&\left. +(2\ell_t(\ell_x^2 + a\ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\psi^2]\right\}\\
&+ 2\frac{\partial}{\partial x}\left\{\theta^2[2\psi_x(\ell_x\psi_x + a\psi_y\ell_y - \psi_t\ell_t) - \ell_x(\psi_x^2 + a\psi_y^2 - \psi_t^2)\right.\\
&\left. +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_x\psi + \ell_x(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\right\}\\
&+ 2a\frac{\partial}{\partial y}\left\{\theta^2[2\psi_y(\ell_x\psi_x + a\psi_y\ell_y) - \ell_y(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_y\psi_t + \ell_y\psi_t^2\right.\\
&\left. +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_y\psi + \ell_y(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\right\}\\
&\geq -8v_t(\ell_{xt}v_x + a\ell_{yt}v_y) + 2(\ell_{xx} + a\ell_{yy} + \ell_{tt} - \zeta)v_t^2\\
&+ 2(\zeta - \frac{\epsilon}{2} - \ell_{xx} - a\ell_{yy} + \ell_{tt})(v_x^2 + av_y^2) + 4(\ell_{xx}v_x^2 + 2a\ell_{xy}v_xv_y + a^2\ell_{yy}v_y^2)\\
&\left\{2A\zeta - 2\left[\frac{\partial}{\partial x}((A+\zeta)\ell_x) + a\frac{\partial}{\partial y}((A+\zeta)\ell_y) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2) + \zeta_{tt}\right\}v^2
\end{aligned} \tag{2.1}
$$

Proof. **2.1.1 Step 1**

Let $v(x,y,t) = \theta(x,y,t)\psi(x,y,t) = e^{\ell(x,y,t)}\psi(x,y,t)$ thus $\psi(x,y,t) = e^{-\ell(x,y,t)}v(x,y,t)$ and by differentiation we have

$$
\begin{cases}
\psi_t = e^{-\ell(x,y,t)}(-\ell_t)v + e^{-\ell(x,y,t)}v_t\\
\psi_{tt} = e^{-\ell(x,y,t)}(\ell_t^2)v + e^{-\ell(x,y,t)}(-\ell_{tt})v + e^{-\ell(x,y,t)}(-\ell_t)v_t + e^{-\ell(x,y,t)}(-\ell_t)v_t + e^{-\ell(x,y,t)}v_{tt}.
\end{cases} \tag{2.2}
$$

Manipulating the results from (2.2) yields

$$\theta\psi_{tt} = e^{\ell(x,y,t)}\psi_{tt} = v_{tt} - 2\ell_t v_t + (\ell_t^2 - \ell_{tt})v. \tag{2.3}$$

Similarly, we have

$$
\begin{cases}
\theta\psi_{xx} = v_{xx} - 2\ell_x v_x + (\ell_x^2 - \ell_{xx})v\\
\theta a\psi_{yy} = a[v_{yy} - 2\ell_y v_y + (\ell_y^2 - \ell_{yy})v].
\end{cases} \tag{2.4}
$$

From (2.3) and (2.4) we obtain

$$\begin{aligned}\theta^2[\psi_{tt}-(\psi_{xx}+a\psi_{yy})]^2 =& e^{2\ell(x,y,t)}[\psi_{tt}-(\psi_{xx}+a\psi_{yy})]^2\\ =& \big\{[v_{tt}-2\ell_t v_t+(\ell_t^2-\ell_{tt})v]-[v_{xx}-2\ell_x v_x+(\ell_x^2-\ell_{xx})v]\\ & -a[v_{yy}-2\ell_y v_y+(\ell_y^2-\ell_{yy})v]\big\}\\ =& |I_1+I_2+I_3|^2\geq 2I_1I_2+2I_2I_3+2I_3I_1\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}I_1 =& v_{tt}-(v_{xx}+av_{yy})+Av\\ I_2 =& -2\ell_t v_t+2\ell_x v_x+2a\ell_y v_y\\ I_3 =& \zeta v\\ A =& (\ell_t^2-\ell_{tt})-(\ell_x^2-\ell_{xx})-a(\ell_y^2-\ell_{yy})-\zeta.\end{aligned} \tag{2.6}$$

2.1.2 Step 2

In Step 2, we shall prove

$$\begin{aligned}2I_1I_2 =& \frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2+Av^2+v_x^2+av_y^2\right)+4v_t(\ell_x v_x+a\ell_y v_y)\right]\\ & -2\left\{\frac{\partial}{\partial x}[2v_x(\ell_x v_x+a\ell_y v_y)-\ell_x(v_x^2+av_y^2)-2\ell_t v_t v_x+\ell_x v_t^2-A\ell_x v^2]\right.\\ & \left.+a\frac{\partial}{\partial y}[2v_y(\ell_x v_x+a\ell_y v_y)-\ell_y(v_x^2+av_y^2)-2\ell_t v_t v_y+\ell_y v_t^2-A\ell_y v^2]\right\}\\ & -8v_t(\ell_{tx}v_x+a\ell_{ty}v_y)+2v_t^2(\ell_{tt}+\ell_{xx}+a\ell_{yy})+4\left[\ell_{xx}v_x^2+2a\ell_{xy}v_xv_y+a^2\ell_{yy}v_y^2\right]\\ & -2(\ell_{xx}+a\ell_{yy}-\ell_{tt})(v_x^2+av_y^2)-2v^2\left[\frac{\partial}{\partial x}(A\ell_x)+a\frac{\partial}{\partial y}(A\ell_y)-\frac{\partial}{\partial t}(A\ell_t)\right].\end{aligned} \tag{2.7}$$

Proof. By direct computation, and substitution from the expressions in (2.6), we have

$$
\begin{aligned}
2I_1I_2 &= 2[v_{tt} - (v_{xx} + av_{yy}) + Av][-2\ell_t v_t + 2\ell_x v_x + 2a\ell_y v_y] \\
&= -4\ell_t v_t v_{tt} + 4v_{tt}\ell_x v_x + 4av_{tt}\ell_y v_y + 4\ell_t v_t(v_{xx} + av_{yy}) \\
&\quad - 4\ell_x v_x(v_{xx} + av_{yy}) - 4a\ell_y v_y(v_{xx} + av_{yy}) \\
&\quad - 4A\ell_t v_t v + 4A\ell_x v_x v + 4aA\ell_y v_y v \\
&= -2\ell_t \frac{\partial}{\partial t}(v_t^2) - 2A\ell_t \frac{\partial}{\partial t}(v^2) + 2A\ell_x \frac{\partial}{\partial x}(v^2) + 2Aa\ell_y \frac{\partial}{\partial y}(v^2) \\
&\quad + \underbrace{4v_{tt}(\ell_x v_x + a\ell_y v_y)}_{1} + \underbrace{4\ell_t v_t(v_{xx} + av_{yy})}_{2} \underbrace{-4(v_{xx} + av_{yy})(\ell_x v_x + a\ell_y v_y)}_{3}.
\end{aligned}
\tag{2.8}
$$

We can next rewrite the last three terms of (2.8), in the order in which they are numbered, as follows:

1.

$$
\begin{aligned}
4v_{tt}(\ell_x v_x + a\ell_y v_y) &= 4\frac{\partial}{\partial t}[v_t(\ell_x v_x + a\ell_y v_y)] - 4v_t \frac{\partial}{\partial t}(\ell_x v_x + a\ell_y v_y) \\
&= 4\frac{\partial}{\partial t}[v_t(\ell_x v_x + a\ell_y v_y)] - 4v_t(\ell_{xt} v_x + \ell_x v_{xt} + a\ell_{yt} v_y + a\ell_y v_y t) \\
&= 4\frac{\partial}{\partial t}[v_t(\ell_x v_x + a\ell_y v_y)] - 4v_t(\ell_{xt} v_x + a\ell_{yt} v_y) - 2\left[\ell_x \frac{\partial}{\partial x}(v_t^2) + a\ell_y \frac{\partial}{\partial y}(v_t^2)\right]
\end{aligned}
$$

2.

$$
\begin{aligned}
4\ell_t v_t(v_{xx} + av_{yy}) &= 4\ell_t v_t v_{xx} + 4a\ell_t v_t v_{yy} \\
&= 4\frac{\partial}{\partial x}(\ell_t v_t v_x) - 4\ell_{tx} v_t v_x - 4\ell_t v_{tx} v_x + 4a\frac{\partial}{\partial y}(\ell_t v_t v_y) - 4a\ell_{ty} v_t v_y - 4a\ell_t v_{ty} v_y \\
&= 4\left[\frac{\partial}{\partial x}(\ell_t v_t v_x) + a\frac{\partial}{\partial y}(\ell_t v_t v_y)\right] - 2\ell_t \frac{\partial}{\partial t}(v_x^2) - 2a\ell_t \frac{\partial}{\partial t}(v_y^2) - 4v_t\ell_{tx} v_x - 4av_t\ell_{ty} v_y \\
&= 4\left[\frac{\partial}{\partial x}(\ell_t v_t v_x) + a\frac{\partial}{\partial y}(\ell_t v_t v_y)\right] - 2\ell_t \frac{\partial}{\partial t}(v_x^2 + av_y^2) - 4v_t(\ell_{tx} v_x + a\ell_{ty} v_y)
\end{aligned}
$$

3.

$$
\begin{aligned}
&-4(v_{xx}+av_{yy})(\ell_x v_x+a\ell_y v_y)\\
&\qquad= -4v_{xx}\ell_x v_x - 4av_{xx}\ell_y v_y - 4av_{yy}\ell_x v_x - 4a^2 v_{yy}\ell_y v_y\\
&\qquad= -4\frac{\partial}{\partial x}(v_x^2\ell_x) + 4v_x^2\ell_{xx} + 4v_x\ell_x v_{xx} - 4a\frac{\partial}{\partial x}(v_x\ell_y v_y) + 4av_x\ell_{yx}v_y + 4av_x\ell_y v_{yx}\\
&\qquad\quad - 4a^2\frac{\partial}{\partial y}(v_y^2\ell_y) + 4a^2 v_y^2\ell_{yy} + 4a^2 v_y\ell_y v_{yy} - 4a\frac{\partial}{\partial y}(v_y\ell_x v_x) + 4av_y\ell_{xy}v_x + 4av_y\ell_x v_{xy}\\
&\qquad= -4\left[\frac{\partial}{\partial x}(v_x^2\ell_x + av_x\ell_y v_y) + \frac{\partial}{\partial y}(av_y\ell_x v_x + a^2 v_y^2\ell_y)\right]\\
&\qquad\quad + 4[\ell_{xx}v_x^2 + a\ell_{yx}v_x v_y + a\ell_{xy}v_x v_y + a^2\ell_{yy}v_y^2]\\
&\qquad\quad + 2\left[\ell_x\frac{\partial}{\partial x}(v_x^2) + a\ell_y\frac{\partial}{\partial y}(v_y^2) + a\ell_x\frac{\partial}{\partial x}(v_y^2) + a^2\ell_y\frac{\partial}{\partial y}(v_y^2)\right]
\end{aligned}
$$

Substituting the expressions derived in 1,2, and 3 into (2.8) we get

$$
\begin{aligned}
2I_1I_2 =& -2\ell_t\frac{\partial}{\partial t}(v_t^2) - 2A\ell_t\frac{\partial}{\partial t}(v^2) + 2A\left[\ell_x\frac{\partial}{\partial x}(v^2) + a\ell_y\frac{\partial}{\partial y}(v^2)\right]\\
&+ 4\frac{\partial}{\partial t}[v_t(\ell_x v_x + a\ell_y v_y)] - 4v_t(\ell_{xt}v_x + a\ell_{yt}vy) - 2\left[\ell_x\frac{\partial}{\partial x}(v_t^2) + a\ell_y\frac{\partial}{\partial y}(v_t^2)\right]\\
&+ 4\left[\frac{\partial}{\partial x}(\ell_t v_t v_x) + a\frac{\partial}{\partial y}(\ell_t v_t v_y)\right] - 2\ell_t\frac{\partial}{\partial t}(v_x^2 + av_y^2) - 4v_t(\ell_{tx}v_x + a\ell_{ty}v_y)\\
&- 4\left[\frac{\partial}{\partial x}(v_x^2\ell_x + av_x\ell_y v_y) + \frac{\partial}{\partial y}(av_y\ell_x v_x + a^2 v_y^2\ell_y)\right]\\
&+ 4[\ell_{xx}v_x^2 + 2a\ell_{xy}v_x v_y + a^2\ell_{yy}v_y^2]\\
&+ 2\left[\ell_x\frac{\partial}{\partial x}(v_x^2) + a\ell_y\frac{\partial}{\partial y}(v_y^2) + a\ell_x\frac{\partial}{\partial x}(v_y^2) + a^2\ell_y\frac{\partial}{\partial y}(v_y^2)\right].
\end{aligned}
$$

Rearranging the terms produces

$$
\begin{aligned}
2I_1I_2 =& -2\ell_t\frac{\partial}{\partial t}(v_t^2) - 2A\ell_t\frac{\partial}{\partial t}(v^2) + 4\frac{\partial}{\partial t}[v_t(\ell_x v_x + a\ell_y v_y)] - 2\ell_t\frac{\partial}{\partial t}(v_x^2 + av_y^2)\\
&+ 2A\left[\ell_x\frac{\partial}{\partial x}(v^2) + a\ell_y\frac{\partial}{\partial y}(v^2)\right] - 2\left[\ell_x\frac{\partial}{\partial x}(v_t^2) + a\ell_y\frac{\partial}{\partial y}(v_t^2)\right] + 4\left[\frac{\partial}{\partial x}(\ell_t v_t v_x) + a\frac{\partial}{\partial y}(\ell_t v_t v_y)\right]\\
&- 4\left[\frac{\partial}{\partial x}(v_x^2\ell_x + av_x\ell_y v_y) + \frac{\partial}{\partial y}(av_y\ell_x v_x + a^2 v_y^2\ell_y)\right]\\
&+ 2\left[\ell_x\frac{\partial}{\partial x}(v_x^2) + a\ell_y\frac{\partial}{\partial y}(v_y^2) + a\ell_x\frac{\partial}{\partial x}(v_y^2) + a^2\ell_y\frac{\partial}{\partial y}(v_y^2)\right]\\
&- 4v_t(\ell_{xt}v_x + a\ell_{yt}v_y) - 4v_t(\ell_{tx}v_x + a\ell_{ty}v_y) + 4[\ell_{xx}v_x^2 + 2a\ell_{xy}v_x v_y + a^2\ell_{yy}v_y^2].
\end{aligned}
\tag{2.9}
$$

Grouping the $\frac{\partial}{\partial t}$ terms in (2.9) yields

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[-2\ell_t v_t^2 - 2A\ell_t v^2 - 2\ell_t(v_x^2 + av_y^2) + 4v_t(\ell_x v_x + a\ell_y v_y)\right] \\
&\qquad + 2\ell_{tt}v_t^2 + 2v^2\frac{\partial}{\partial t}(A\ell_t) + 2\ell_{tt}(v_x^2 + av_y^2).
\end{aligned} \tag{2.10}$$

Grouping the $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ terms we have

$$\begin{aligned}
&\frac{\partial}{\partial x}[2A\ell_x v^2 - 2\ell_x v_t^2 + 4\ell_t v_t v_x - 4v_x(\ell_x v_x + a\ell_y v_y) + 2\ell_x(v_x^2 + av_y^2)] \\
&\quad + a\frac{\partial}{\partial y}[2A\ell_y v^2 - 2\ell_y v_t^2 + 4\ell_t v_t v_y - 4v_y(\ell_x v_x + a\ell_y v_y) + 2\ell_y(v_x^2 + av_y^2)] \\
&\quad - 2v^2\left[\frac{\partial}{\partial x}(A\ell_x) + a\frac{\partial}{\partial y}(A\ell_y)\right] + 2v_t^2(\ell_{xx} + a\ell_{yy}) - 2(\ell_{xx} + a\ell_{yy})(v_x^2 + av_y^2).
\end{aligned} \tag{2.11}$$

Finally, substituting (2.10) and (2.11) into (2.9) and rearranging the terms gives the result in (2.7).

□

2.1.3 Step 3

Applying the substitutions in (2.6) we shall prove for all $\epsilon > 0$

$$\begin{aligned}
2I_1I_3 = &\frac{\partial}{\partial t}(2\zeta v v_t - \zeta_t v^2) + (\zeta_{tt} + 2A\zeta)v^2 - 2\zeta v_t^2 \\
&- 2\left[\frac{\partial}{\partial x}(\zeta v_x v) + a\frac{\partial}{\partial y}(\zeta v_y v)\right] + 2v(\zeta_x v_x + a\zeta_y v_y) + 2\zeta(v_x^2 + av_y^2) \\
\geq &\frac{\partial}{\partial t}(2\zeta v v_t - \zeta_t v^2) + \left[\zeta_{tt} + 2A\zeta - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2)\right]v^2 - 2\zeta v_t^2 \\
&+ (2\zeta - \epsilon)(v_x^2 + av_y^2) - 2\left[\frac{\partial}{\partial x}(\zeta v_x v) + a\frac{\partial}{\partial y}(\zeta v_y v)\right].
\end{aligned} \tag{2.12}$$

Proof.

$$\begin{aligned}
2I_1I_3 = &2[v_{tt} - (v_{xx} + av_{yy}) + Av]\zeta v \\
= &2\zeta v v_{tt} - 2\zeta v(v_{xx} + av_{yy}) + 2A\zeta v^2.
\end{aligned} \tag{2.13}$$

Where

$$\begin{aligned}
2\zeta v v_{tt} &= 2\frac{\partial}{\partial t}(\zeta v v_t) - 2v_t\frac{\partial}{\partial t}(\zeta v) \\
&= 2\frac{\partial}{\partial t}(\zeta v v_t) - 2v_t\zeta_t v - 2v_t^2\zeta \\
&= 2\frac{\partial}{\partial t}(\zeta v v_t) - \frac{\partial}{\partial t}(\zeta_t v^2) + \zeta_{tt}v^2 - 2v_t^2\zeta
\end{aligned}$$

$$\begin{aligned}
-2\zeta v(v_{xx} + av_{yy}) &= -2\zeta v v_{xx} - 2a\zeta v v_{yy} \\
&= -2\frac{\partial}{\partial x}(\zeta v v_x) + 2v_x\frac{\partial}{\partial x}(\zeta v) - 2a\frac{\partial}{\partial y}(\zeta v v_y) + 2av_y\frac{\partial}{\partial y}(\zeta v) \\
&= -2\left[\frac{\partial}{\partial x}(\zeta v v_x) + a\frac{\partial}{\partial y}(\zeta v v_y)\right] + 2\zeta_x v_x v + 2a\zeta_y v_y v + 2\zeta v_x^2 + 2a\zeta v_y^2 \\
&= -2\left[\frac{\partial}{\partial x}(\zeta v v_x) + a\frac{\partial}{\partial y}(\zeta v v_y)\right] + 2v(\zeta_x v_x + a\zeta_y v_y) + 2\zeta(v_x^2 + av_y^2).
\end{aligned}$$

Substituting into (2.13) and using $2v(\zeta_x v_x + a\zeta_y v_y) \geq -\epsilon(v_x^2 + av_y^2) - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2)v^2$, where $\epsilon > 0$ is arbitrary and recalling $a > 0$, we have the result in (2.12). □

2.1.4 Step 4

In Step 4 we shall prove that

$$2I_2I_3 = \frac{\partial}{\partial t}(-2\zeta\ell_t v^2) + \frac{\partial}{\partial x}(2\zeta\ell_x v^2) + \frac{\partial}{\partial y}(2a\zeta\ell_y v^2) + 2v^2\left[\frac{\partial}{\partial t}(\ell_t\zeta) - \frac{\partial}{\partial x}(\ell_x\zeta) - a\frac{\partial}{\partial y}(\ell_y\zeta)\right] \quad (2.14)$$

Proof. By applying the definitions in (2.6) we have

$$\begin{aligned}
2I_2I_3 &= 2\left(-2\ell_t v_t + 2\ell_x v_x + 2a\ell_y v_y\right)\zeta v \\
&= -2\ell_t\zeta\frac{\partial}{\partial t}(v^2) + 2\zeta\ell_x\frac{\partial}{\partial x}(v^2) + 2a\zeta\ell_y\frac{\partial}{\partial y}(v^2)
\end{aligned} \quad (2.15)$$

Where

$$\begin{aligned}
-2\ell_t\zeta\frac{\partial}{\partial t}(v^2) &= \frac{\partial}{\partial t}(-2\ell_t\zeta v^2) + 2\ell_{tt}\zeta v^2 + 2\ell_t\zeta_t v^2 \\
&= \frac{\partial}{\partial t}(-2\ell_t\zeta v^2) + 2v^2(\ell_{tt}\zeta + \ell_t\zeta_t) \\
&= \frac{\partial}{\partial t}(-2\ell_t\zeta v^2) + 2v^2\frac{\partial}{\partial t}(\ell_t\zeta)
\end{aligned}$$

$$\begin{aligned}
2\zeta\ell_x\frac{\partial}{\partial x}(v^2) + 2a\zeta\ell_y\frac{\partial}{\partial y}(v^2) &= \frac{\partial}{\partial x}(2\zeta\ell_x v^2) - 2\ell_{xx}\zeta v^2 - 2\ell_x\zeta_x v^2 \\
&\quad + \frac{\partial}{\partial y}(2a\zeta\ell_y v^2) - 2a\ell_{yy}\zeta v^2 - 2a\ell_y\zeta_y v^2 \\
&= \frac{\partial}{\partial x}(2\zeta\ell_x v^2) + \frac{\partial}{\partial y}(2a\zeta\ell_y v^2) - 2v^2\frac{\partial}{\partial x}(\ell_x\zeta) - 2v^2\frac{\partial}{\partial y}(a\ell_y\zeta)
\end{aligned}$$

Substituting these results into (2.15) gives the result in (2.14). □

2.1.5 Step 5

Making the appropriate substitutions into (2.5), we will prove in Step 5 that

$$\begin{aligned}
&\theta^2[\psi_{tt} - (\psi_{xx} + a\psi_{yy})]^2 \\
&\geq \frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2 + v_x^2 + av_y^2\right) + 4v_t(\ell_x v_x + a\ell_y v_y) + 2\zeta v v_t - 2\ell_t(A+\zeta)v^2 - \zeta_t v^2\right] \\
&- 2\frac{\partial}{\partial x}\left[2v_x(v_x\ell_x + av_y\ell_y) - \ell_x(v_x^2 + av_y^2) - 2\ell_t v_t v_x + \ell_x v_t^2 + \zeta v v_x - (A+\zeta)\ell_x v^2\right] \\
&- 2a\frac{\partial}{\partial y}\left[2v_y(v_x\ell_x + av_y\ell_y) - \ell_y(v_x^2 + av_y^2) - 2\ell_t v_t v_y + \ell_y v_t^2 + \zeta v v_y - (A+\zeta)\ell_y v^2\right] \\
&- 8v_t(\ell_{xt}v_x + a\ell_{yt}v_y) + 2(\ell_{xx} + a\ell_{yy} + \ell_{tt} - \zeta)v_t^2 \\
&+ 2(\zeta - \frac{\epsilon}{2} - \ell_{xx} - a\ell_{yy} + \ell_{tt})(v_x^2 + av_y^2) + 4(\ell_{xx}v_x^2 + 2a\ell_{xy}v_x v_y + a^2\ell_{yy}v_y^2) \\
&+ \left\{2A\zeta - 2\left[\frac{\partial}{\partial x}((A+\zeta)\ell_x) + a\frac{\partial}{\partial y}((A+\zeta)\ell_y) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2) + \zeta_{tt}\right\}v^2
\end{aligned} \tag{2.16}$$

Proof. Substituting (2.7) for $2I_1I_2$, (2.12) for $2I_1I_3$, and (2.14) for $2I_2I_3$ we obtain the inequality

$$
\begin{aligned}
&e^{2\ell(x,y,t)}[\psi_{tt} - (\psi_{xx} + a\psi_{yy})]^2 \\
&\geq \frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2 + Av^2 + v_x^2 + av_y^2\right) + 4v_t(\ell_x v_x + a\ell_y v_y)\right] \\
&\quad - 2\left\{\frac{\partial}{\partial x}[2v_x(\ell_x v_x + a\ell_y v_y) - \ell_x(v_x^2 + av_y^2) - 2\ell_t v_t v_x + \ell_x v_t^2 - A\ell_x v^2]\right. \\
&\quad \left. + a\frac{\partial}{\partial y}[2v_y(\ell_x v_x + a\ell_y v_y) - \ell_y(v_x^2 + av_y^2) - 2\ell_t v_t v_y + \ell_y v_t^2 - A\ell_y v^2]\right\} \\
&\quad - 8v_t(\ell_{tx}v_x + a\ell_{ty}v_y) + 2v_t^2(\ell_{tt} + \ell_{xx} + a\ell_{yy}) + 4\left[\ell_{xx}v_x^2 + 2a\ell_{xy}v_x v_y + a^2\ell_{yy}v_y^2\right] \\
&\quad - 2(\ell_{xx} + a\ell_{yy} - \ell_{tt})(v_x^2 + av_y^2) - 2v^2\left[\frac{\partial}{\partial x}(A\ell_x) + a\frac{\partial}{\partial y}(A\ell_y) - \frac{\partial}{\partial t}(A\ell_t)\right] \\
&\quad + \frac{\partial}{\partial t}(2\zeta v v_t - \zeta_t v^2) + \left[\zeta_{tt} + 2A\zeta - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2)\right]v^2 - 2\zeta v_t^2 \\
&\quad + (2\zeta - \epsilon)(v_x^2 + av_y^2) - 2\left[\frac{\partial}{\partial x}(\zeta v_x v) + a\frac{\partial}{\partial y}(\zeta v_y v)\right] \\
&\quad + \frac{\partial}{\partial t}(-2\zeta\ell_t v^2) + \frac{\partial}{\partial x}(2\zeta\ell_x v^2) + \frac{\partial}{\partial y}(2a\zeta\ell_y v^2) \\
&\quad + 2v^2\left[\frac{\partial}{\partial t}(\ell_t\zeta) - \frac{\partial}{\partial x}(\ell_x\zeta) - a\frac{\partial}{\partial y}(\ell_y\zeta)\right].
\end{aligned}
$$

Combining the $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ terms, the $(v_x^2 + av_y^2)$ terms, and the v^2 and v_t^2 terms we obtain the result in (2.16). □

2.1.6 Step 6

We particularize (2.16) with

$$
v = \theta\psi = e^{\ell}\psi \Rightarrow \begin{cases} v_t = \theta(\psi_t + \ell_t\psi),\ v_x = \theta(\psi_x + \ell_x\psi),\ v_y = \theta(\psi_y + \ell_y\psi) \\ v_x^2 + av_y^2 = \theta^2[(\psi_x^2 + \ell_x\psi)^2 + a(\psi_y + \ell_y\psi)^2]. \end{cases}
$$

Hence we shall prove the terms under $\frac{\partial}{\partial t}$ in (2.16) become

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2 + v_x^2 + av_y^2\right) + 4v_t(\ell_x v_x + a\ell_y v_y) + 2\zeta v v_t - 2\ell_t(A+\zeta)v^2 - \zeta_t v^2\right] \\
&= \frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(\psi_t^2 + \psi_x^2 + a\psi_y^2) + 4\psi_t(\ell_x\psi_x + a\ell_y\psi_y) + 2(2\ell_x^2 + 2a\ell_y^2 - 2\ell_t^2 + \zeta)\psi\psi_t \right. \\
&\quad \left. + (2\ell_t(\ell_x^2 + a\ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\psi^2]\right\}
\end{aligned}
\tag{2.17}
$$

Proof. Making the relevant substitutions we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left[-2\ell_t\left(v_t^2+v_x^2+av_y^2\right)+4v_t(\ell_x v_x+a\ell_y v_y)+2\zeta v v_t-2\ell_t(A+\zeta)v^2-\zeta_t v^2\right]\\
&\quad=\frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t((\psi_t+\ell_t\psi)^2+(\psi_x+\ell_x\psi)^2)+a(\psi_y+\ell_y\psi)^2+\right.\\
&\qquad\qquad+4(\ell_x(\psi_x+\ell_x\psi)(\psi_t+\ell_t\psi)+a\ell_y(\psi_y+\ell_y\psi)(\psi_t+\ell_t\psi))\\
&\qquad\qquad\qquad\left.+2\zeta\psi(\psi_t+\ell_t\psi)-2\ell_t(A+\zeta)\psi^2-\zeta_t\psi^2]\right\}\\
&\quad=\frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(\psi_t^2+2\ell_t\psi\psi_t+\ell_t^2\psi^2+\psi_x^2+2\ell_x\psi\psi_x+\ell_x^2\psi^2+a\psi_y^2+2a\ell_y\psi\psi_y+a\ell_y^2\psi^2)\right.\\
&\qquad+4(\ell_x\psi_x\psi_t+\ell_x\ell_t\psi\psi_x+\ell_x^2\psi\psi_t+\ell_x^2\ell_t\psi^2+a\ell_y\psi_y\psi_t+a\ell_y\ell_t\psi\psi_y+a\ell^2\psi\psi_t+a\ell_y^2\psi\psi_t+a\ell_y^2\ell_t\psi^2)\\
&\qquad\qquad\left.+2\zeta\psi\psi_t+2\zeta\ell_t\psi^2-2\ell_t A\psi^2-2\ell_t\zeta\psi^2-\zeta_t\psi^2]\right\}\\
&\quad=\frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(\psi_t^2+\psi_x^2+a\psi_y^2)+4\psi_t(\ell_x\psi_x+a\ell_y\psi_y)+2(2\ell_x^2+2a\ell_y^2-2\ell_t^2+\zeta)\psi\psi_t\right.\\
&\qquad\left.+(2\ell_t(\ell_x^2+a\ell_y^2)-2\ell_t^3-2A\ell_t-\zeta_t)\psi^2]\right\}
\end{aligned}
$$

which is the desired result in (2.17). □

2.1.7 Step 7

With the specialization $v=\theta\psi$ the $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ terms in (2.16) become

$$
\begin{aligned}
&-2\frac{\partial}{\partial x}\left\{\theta^2[2\psi_x(\ell_x\psi_x+a\psi_y\ell_y)-\ell_x(\psi_x^2+a\psi_y^2)-2\ell_t\psi_x\psi_t+\ell_x\psi_t^2\right.\\
&\qquad\qquad\left.+2\left(\ell_x^2+a\ell_y^2-\ell_t^2+\frac{\zeta}{2}\right)\psi_x\psi+\ell_x(\ell_x^2+a\ell_y^2-\ell_t^2-A)\psi^2]\right\},\\
&-2a\frac{\partial}{\partial y}\left\{\theta^2[2\psi_y(\ell_x\psi_x+a\psi_y\ell_y)-\ell_y(\psi_x^2+a\psi_y^2)-2\ell_t\psi_y\psi_t+\ell_y\psi_t^2\right.\\
&\qquad\qquad\left.+2\left(\ell_x^2+a\ell_y^2-\ell_t^2+\frac{\zeta}{2}\right)\psi_y\psi+\ell_y(\ell_x^2+a\ell_y^2-\ell_t^2-A)\psi^2]\right\}
\end{aligned}
\tag{2.18}
$$

Proof. From the appropriate substitutions the $\frac{\partial}{\partial x}$ terms are

$$
\begin{aligned}
&-2\frac{\partial}{\partial x}\left[2v_x(v_x\ell_x + av_y\ell_y) - \ell_x(v_x^2 + av_y^2) - 2\ell_t v_t v_x + \ell_x v_t^2 + \zeta v v_x - (A+\zeta)\ell_x v^2\right] \\
&\quad = -2\frac{\partial}{\partial x}\left\{2\theta^2(\ell_x\psi + \psi_x)[\psi(\ell_x^2 + a\ell_y^2) + (\ell_x\psi_x + a\ell_y\psi_y)] - \ell_x[\theta^2(\ell_x\psi+\psi_x)^2 + a\theta^2(\psi_y + \ell_y\psi)^2]\right. \\
&\qquad \left. -2\ell_t\theta^2(\ell_x\psi+\psi_x)(\ell_t\psi+\psi_t) + \ell_x\theta^2(\ell_t\psi+\psi_t)^2 + \zeta\theta^2\psi(\ell_x\psi+\psi_x) - (A+\zeta)\ell_x\theta^2\psi^2\right\} \\
&\quad = -2\frac{\partial}{\partial x}\left\{2\theta^2\psi_x(\ell_x\psi_x + a\ell_y\psi_y) + 2\theta^2\psi_x\psi(\ell_x^2 + a\ell_y^2) + 2\theta^2\ell_x\psi(\ell_x\psi_x + a\ell_y\psi_y)\right. \\
&\qquad + 2\theta^2\ell_x\psi^2(\ell_x^2 + a\ell_y^2) - \ell_x\theta^2\psi_x^2 - \ell_x\theta^2\ell_x^2\psi^2 - 2\ell_x^2\theta^2\psi_x\psi - a\ell_x\theta^2\psi_y^2 - a\ell_x\theta^2\ell_y^2\psi^2 \\
&\qquad - 2a\ell_x\theta^2\ell_x\ell_y\psi_y\psi - 2\ell_t\theta^2\psi_t\psi - 2\ell_t\theta^2\psi_t\psi - 2\ell_t^2\theta^2\psi\psi_x - 2\ell_t^2\ell_x\theta^2\psi^2 + \ell_x\theta^2\psi_t^2 + \ell_x\theta^2\ell_t^2\psi^2 \\
&\qquad \left. +2\theta^2\ell_x\ell_t\psi\psi_t + \zeta\theta^2\psi_x\psi + \zeta\theta^2\ell_x\psi^2 - (A+\zeta)\ell_x\theta^2\psi^2\right\} \\
&\quad = -2\frac{\partial}{\partial x}\left\{\theta^2[2\psi_x(\ell_x\psi_x + a\psi_y\ell_y) - \ell_x(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_x\psi_t + \ell_x\psi_t^2\right. \\
&\qquad \left. +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_x\psi + \ell_x(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\right\}
\end{aligned}
$$

which is the desired result. Similarly, we have the $\frac{\partial}{\partial y}$ terms simplify as follows

$$
\begin{aligned}
&-2\frac{\partial}{\partial y}\left[2av_y(v_x\ell_x + av_y\ell_y) - a\ell_y(v_x^2 + av_y^2) - 2a\ell_t v_t v_y + a\ell_y v_t^2 + a\zeta v v_y - (A+\zeta)a\ell_y v^2\right] \\
&\quad = -2\frac{\partial}{\partial y}\left\{2a\theta^2(\ell_y\psi + \psi_y)[\psi(\ell_x^2 + a\ell_y^2) + (\ell_x\psi_x + a\ell_y\psi_y)] - a\ell_y[\theta^2(\ell_x\psi+\psi_x)^2 + a\theta^2(\psi_y + \ell_y\psi)^2]\right. \\
&\qquad \left. -2a\ell_t\theta^2(\ell_y\psi+\psi_y)(\ell_t\psi+\psi_t) + a\ell_y\theta^2(\ell_t\psi+\psi_t)^2 + a\zeta\theta^2\psi(\ell_y\psi+\psi_y) - (A+\zeta)\ell_y\theta^2\psi^2\right\} \\
&\quad = -2\frac{\partial}{\partial y}\left\{2a\theta^2\psi_y(\ell_x\psi_x + a\ell_y\psi_y) + 2a\theta^2\psi_y\psi(\ell_x^2 + a\ell_y^2) + 2a\theta^2\ell_y\psi(\ell_x\psi_x + a\ell_y\psi_y)\right. \\
&\qquad + 2a\theta^2\ell_y\psi^2(\ell_x^2 + a\ell_y^2) - a\ell_y\theta^2\psi_x^2 - a\ell_y\theta^2\ell_x^2\psi^2 - 2a^2\ell_y^2\theta^2\psi_y\psi - a^2\ell_y\theta^2\psi_y^2 - a^2\ell_y\theta^2\ell_y^2\psi^2 \\
&\qquad - 2a\ell_y\theta^2\ell_x\psi_x\psi - 2a\ell_t\theta^2\psi_t\psi_y - 2a\ell_t\theta^2\psi_t\psi\ell_y - 2a\ell_t^2\theta^2\psi\psi_y - 2a\theta^2\ell_t^2\ell_y\psi^2 + a\ell_y\theta^2\psi_t^2 + a\ell_y\theta^2\ell_t^2\psi^2 \\
&\qquad \left. +2a\theta^2\ell_y\ell_t\psi\psi_t + a\zeta\theta^2\psi_y\psi + a\zeta\theta^2\ell_y\psi^2 - (A+\zeta)a\ell_y\theta^2\psi^2\right\} \\
&\quad = -2a\frac{\partial}{\partial y}\left\{\theta^2[2\psi_y(\ell_x\psi_x + a\psi_y\ell_y) - \ell_y(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_y\psi_t + \ell_y\psi_t^2\right. \\
&\qquad \left. +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_y\psi + \ell_y(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\right\}
\end{aligned}
$$

□

2.1.8 Step 8

Finally, inserting the expressions (2.17) and (2.18) into (2.16) we have

$$
\begin{aligned}
\theta^2[\psi_{tt} - (\psi_{xx} + a\psi_{yy})]^2 \geq \frac{\partial}{\partial t} \{ & \theta^2[-2\ell_t(\psi_t^2 + \psi_x^2 + a\psi_y^2) + 4\psi_t(\ell_x\psi_x + a\ell_y\psi_y) + 2(2\ell_x^2 + 2a\ell_y^2 - 2\ell_t^2 + \zeta)\psi\psi_t \\
& +(2\ell_t(\ell_x^2 + a\ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\psi^2]\} \\
& - 2\frac{\partial}{\partial x}\{\theta^2[2\psi_x(\ell_x\psi_x + a\psi_y\ell_y) - \ell_x(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_x\psi_t + \ell_x\psi_t^2 \\
& \qquad +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_x\psi + \ell_x(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\Big\} \\
& - 2a\frac{\partial}{\partial y}\{\theta^2[2\psi_y(\ell_x\psi_x + a\psi_y\ell_y) - \ell_y(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_y\psi_t + \ell_y\psi_t^2 \\
& \qquad +2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_y\psi + \ell_y(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2]\Big\} \\
& - 8v_t(\ell_{xt}v_x + a\ell_{yt}v_y) + 2(\ell_{xx} + a\ell_{yy} + \ell_{tt} - \zeta)v_t^2 \\
& + 2(\zeta - \frac{\epsilon}{2} - \ell_{xx} - a\ell_{yy} + \ell_{tt})(v_x^2 + av_y^2) + 4(\ell_{xx}v_x^2 + 2a\ell_{xy}v_xv_y + a^2\ell_{yy}v_y^2) \\
& \left\{2A\zeta - 2\left[\frac{\partial}{\partial x}((A+\zeta)\ell_x) + a\frac{\partial}{\partial y}((A+\zeta)\ell_y) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2) + \zeta_{tt}\right\}v^2
\end{aligned}
$$

which is the desired result. Thus we have completed the proof to Lemma 1. □

2.2 Statement of Lemma as applied to equation (1.4)

Similarly as the assumptions in Lemma 1, let

$$\phi(x,y,t) \in C^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t);\ \ell(x,y,t) \in C^3(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t);\ \zeta(x,y,t) \in C^2 \text{ in } t \text{ and } C^1 \text{ in } x,y$$

be given and set $\theta(x,y,t) = e^{\ell(x,y,t)}$. Additionally, set

$$
\begin{aligned}
v(x,y,t) &= \theta(x,y,t)\phi(x,y,t) \\
A &= (\ell_t^2 - \ell_{tt}) - a(\ell_x^2 - \ell_{xx}) - (\ell_y^2 - \ell_{yy}).
\end{aligned}
$$

Then, letting $\epsilon > 0$ be arbitrary, we have the following pointwise inequality (for ease of computation, we use the substitution $a = \frac{1-\mu}{2}$ throughout the statement and proof of the lemma)

$$\begin{aligned}
&\theta^2[\phi_{tt} - (a\phi_{xx} + \phi_{yy})]^2 - \frac{\partial}{\partial t}\left\{\theta^2[-2\ell_t(\phi_t^2 + a\phi_x^2 + \phi_y^2) + 4\psi_t(a\ell_x\phi_x + \ell_y\phi_y) + 2(2a\ell_x^2 + 2\ell_y^2 - 2\ell_t^2 + \zeta)\phi\phi_t\right. \\
&\left. +(2\ell_t(a\ell_x^2 + \ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\phi^2]\right\} \\
&+ 2a\frac{\partial}{\partial x}\left\{\theta^2[2\phi_x(a\ell_x\phi_x + \phi_y\ell_y - \phi_t\ell_t) - \ell_x(a\phi_x^2 + \phi_y^2 - \phi_t^2)\right. \\
&\left. +2\left(a\ell_x^2 + \ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\phi_x\phi + \ell_x(a\ell_x^2 + \ell_y^2 - \ell_t^2 - A)\phi^2]\right\} \\
&+ 2\frac{\partial}{\partial y}\left\{\theta^2[2\phi_y(a\ell_x\phi_x + \phi_y\ell_y - \phi_t\ell_t) - \ell_y(a\phi_x^2 + \phi_y^2 - \phi_t^2)\right. \\
&\left. +2\left(a\ell_x^2 + \ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\phi_y\phi + \ell_y(a\ell_x^2 + \ell_y^2 - \ell_t^2 - A)\phi^2]\right\} \\
&\qquad\qquad \geq -8v_t(a\ell_{xt}v_x + \ell_{yt}v_y) + 2(a\ell_{xx} + \ell_{yy} + \ell_{tt} - \zeta)v_t^2 \\
&\qquad\qquad + 2(\zeta - \frac{\epsilon}{2} - a\ell_{xx} - \ell_{yy} + \ell_{tt})(av_x^2 + v_y^2) + 4(a^2\ell_{xx}v_x^2 + 2a\ell_{xy}v_xv_y + \ell_{yy}v_y^2) \\
&\qquad\qquad + \left\{2A\zeta - 2\left[a\frac{\partial}{\partial x}((A+\zeta)\ell_x) + \frac{\partial}{\partial y}((A+\zeta)\ell_y) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{1}{\epsilon}(a\zeta_x^2 + \zeta_y^2) + \zeta_{tt}\right\}v^2
\end{aligned} \tag{2.19}$$

Chapter 3

Carleman Estimate

This chapter includes the resulting Carleman Estimate for smooth solutions to equation (1.3) as well as establishing basic assumptions and inequalities necessary for the estimate. A similar estimate is also established for (1.4), which, when combining with the estimates for (1.2) and (1.3) will enable a final estimate for the entire model. Only the development of the results for (1.3) is included in detail since the derivation of estimates for (1.4) follow a similar process.

3.1 Basic Assumptions

For convenience, we simplify the inequality in (2.1) using the following substitutions.

$$\begin{aligned}
M =& \theta^2[-2\ell_t(\psi_t^2 + \psi_x^2 + a\psi_y^2) + 4\psi_t(\ell_x\psi_x + a\ell_y\psi_y) + 2(2\ell_x^2 + 2a\ell_y^2 - 2\ell_t^2 + \zeta)\psi\psi_t \\
&+ (2\ell_t(\ell_x^2 + a\ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\psi^2] \\
V_1 =& 2\theta^2[2\psi_x(\ell_x\psi_x + a\psi_y\ell_y - \psi_t\ell_t) - \ell_x(\psi_x^2 + a\psi_y^2 - \psi_t^2) \\
&+ 2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_x\psi + \ell_x(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2] \\
V_2 =& 2\theta^2[2\psi_y(\ell_x\psi_x + a\psi_y\ell_y) - \ell_y(\psi_x^2 + a\psi_y^2) - 2\ell_t\psi_y\psi_t + \ell_y\psi_t^2 \\
&+ 2\left(\ell_x^2 + a\ell_y^2 - \ell_t^2 + \frac{\zeta}{2}\right)\psi_y\psi + \ell_y(\ell_x^2 + a\ell_y^2 - \ell_t^2 - A)\psi^2] \\
\tilde{B} =& 2A\zeta - 2\left[\frac{\partial}{\partial x}((A+\zeta)\ell_x) + a\frac{\partial}{\partial y}((A+\zeta)\ell_y) - \frac{\partial}{\partial t}((A+\zeta)\ell_t)\right] - \frac{1}{\epsilon}(\zeta_x^2 + a\zeta_y^2) + \zeta_{tt}
\end{aligned}$$

Thus, (2.1) simplifies to be

$$\begin{aligned}
\theta^2[\psi_{tt} - (\psi_{xx} + a\psi_{yy})]^2 - \frac{\partial}{\partial t}M + \frac{\partial}{\partial x}V_1 + a\frac{\partial}{\partial y}V_2 \\
&\geq -8v_t(\ell_{xt}v_x + a\ell_{yt}v_y) + 2(\ell_{xx} + a\ell_{yy} + \ell_{tt} - \zeta)v_t^2 \\
&+ 2(\zeta - \frac{\epsilon}{2} - \ell_{xx} - a\ell_{yy} + \ell_{tt})(v_x^2 + av_y^2) + 4(\ell_{xx}v_x^2 + 2a\ell_{xy}v_xv_y + a^2\ell_{yy}v_y^2) \\
&+ \tilde{B}v^2
\end{aligned} \tag{3.1}$$

Consider the convex function $d(x, y) = (x - x_0)^2 + (y - y_0)^2$ where (x_0, y_0) is a fixed point outside $\bar{\Omega}$. Then for the parameter $\tau > 0$ and constant α we define the psuedo-convex function P, ℓ and ζ as follows

$$\begin{aligned}
P(x, y, t) &\equiv \left[(x - x_0)^2 + (y - y_0)^2 - ac\left(t - \frac{T}{2}\right)^2 \right] \qquad 0 \leq t \leq T, (x, y) \in \Omega, \\
\ell(x, y, t) &\equiv \tau P(x, y, t), \\
\zeta &\equiv \tau\alpha.
\end{aligned}$$

Where $T > 0$ and $0 < c < 1$ are selected following the specified criterion below where we first define T_0 by the equivalence

$$T_0^2 \equiv \frac{4}{a} \max_{(x,y)\in\bar{\Omega}} \left\{ (x - x_0)^2 + (y - y_0)^2 \right\}.$$

Thus for $T > T_0$ there exists $\delta > 0$ satisfying (recall a is as defined in (2.1))

$$T^2 > T_0^2 + \frac{4\delta}{a}$$

Then for this δ there exists a constant c such that $0 < c < 1$ and

$$acT^2 > 4 \max_{(x,y)\in\bar{\Omega}} \left\{ (x - x_0)^2 + (y - y_0)^2 \right\} + 4\delta$$

holds. Thus the c and T in the definition of ℓ are chosen in this way and P exhibits the following properties for the given δ:

- $P(x, y, 0) \equiv P(x, y, T) = d(x, y) - ac\frac{T^2}{4} \leq \max_{\bar{\Omega}} d(x, y) - ac\frac{T^2}{4} \leq -\delta, \quad \forall (x, y) \in \Omega$

- There exists a small neighborhood around $\frac{T}{2}$, say $\frac{T}{2} \in (t_0, t_1) \subset (0, T)$, such that

$$\min_{(x,y)\in\bar{\Omega}, t\in[t_0,t_1]} P(x,y,t) \geq \sigma > 0$$

where $\sigma < \min_{\bar{\Omega}} d(x,y) \leq d(x,y) = P(x,y,\frac{T}{2})$. Then we have the definition

$$\min_{\bar{\Omega}} d(x,y) - ac\left(t_1 - \frac{T}{2}\right)^2 \equiv \sigma > 0, \text{ so } t_1 - \frac{T}{2} = \sqrt{\frac{\min_{\bar{\Omega}} d(x,y) - \sigma}{ac}}.$$

From this we can define the following region

$$Q(\sigma) \equiv \{(x,y,t) : (x,y) \in \Omega, 0 \leq t \leq T, P(x,y,t) \geq \sigma > 0\}$$

which will be used to separate $\Omega \times [0,T]$ since certain pointwise inequalities, derived later, will only hold on $Q(\sigma)$. Next, applying the substitutions from Lemma 1 we have

$$\theta(x,y,t) = e^{\ell(x,y,t)} = e^{\tau P(x,y,t)}.$$

Moreover, as a result of these choices we have the following specializations of Lemma 1:

$$\begin{aligned} \ell_x &= 2\tau(x-x_0) & \ell_x^2 + a\ell_y^2 &= 4\tau^2\{(x-x_0)+a(y-y_0)\} \\ \ell_y &= 2\tau(y-y_0) & \ell_{xx} + a\ell_{yy} &= 2\tau(1+a) \\ \ell_{xx} &= \ell_{yy} = 2\tau & \ell_{xy} &= \ell_{yx} = 0 \end{aligned} \tag{3.2}$$

Continuing with partial derivatives of ℓ with respect to t we have the specializations

$$\ell_t = -2ac\tau\left(t - \frac{T}{2}\right), \qquad \ell_{tt} = -2ac\tau, \qquad \ell_{tx} = \ell_{ty} = 0 \tag{3.3}$$

The partial derivatives of ζ then become

$$\zeta_t = 0, \qquad \zeta_x + \zeta_y = \tau(\alpha_x + \alpha_y) = 0 \tag{3.4}$$

3.2 A resulting pointwise inequality

Theorem 3.2.1. *The aforementioned specializations and definitions thus make the pointwise estimate in* (2.1)*:*

$$\theta^2 \left[\psi_{tt} - (\psi_{xx} + a\psi_{yy})\right]^2 - \frac{\partial}{\partial t}M + \frac{\partial}{\partial x}V_1 + a\frac{\partial}{\partial y}V_2$$
$$\geq 2\tau[(2+2a) - 2ac - \alpha]v_t^2 + 2\tau\left[\alpha - \frac{\epsilon}{2\tau} - (2+2a) - 2ac\right](v_x^2 + av_y^2) + 4\tau\left(2v_x^2 + 2a^2v_y^2\right) + \theta^2\tilde{B}\psi^2 \tag{3.5}$$

where A *and* $\tilde{B}$ *are:*

$$A = \tau^2\left[4a^2c^2\left(t - \frac{T}{2}\right)^2 - \left(4(x-x_0)^2 + 4a(y-y_0)^2\right)\right] + \tau[2ac + (2+2a) - \alpha] \tag{3.6}$$

$$\begin{aligned}\tilde{B} =& 2\tau^3\left\{4[2ac + (2+2a) - \alpha]\left((x-x_0)^2 + a(y-y_0)^2\right) + 16(x-x_0)^2 + 16a^2(y-y_0)^2\right.\\ &\left.-[6ac + (2+2a) - \alpha]4a^2c^2\left(t - \frac{T}{2}\right)^2\right\} + O(\tau^2).\end{aligned} \tag{3.7}$$

Proof. By direct computation we have

$$\begin{aligned}A =& (\ell_t^2 - \ell_{tt}) - (\ell_x^2 - \ell_{xx}) - a(\ell_y^2 - \ell_{yy}) - \zeta\\ =& \left[4a^2c^2\tau^2\left(t - \frac{T}{2}\right)^2 + 2ac\tau\right] - [\tau^2d_x^2 - \tau d_{xx}] - a[\tau^2 d_y^2 - \tau d_{yy}] - \tau\alpha\\ =& \tau^2\left[4a^2c^2\left(t - \frac{T^2}{2}\right)^2 - (d_x^2 + ad_y^2)\right] + \tau[2ac + (d_{xx} + ad_{yy}) - \alpha]\end{aligned}$$

Substituting the partial derivatives of $d(x, y)$ yields the result in (3.6). Now, recalling $\zeta = \tau\alpha$ we obtain (via (3.2), (3.3), and (3.4))

$$
\begin{aligned}
2A\zeta =& 2\tau^3\alpha\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-(d_x^2+ad_y^2)\right]+2\tau^2\alpha[2c+(d_{xx}+ad_{yy})-\alpha] \\
=& 2\tau^3\alpha\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]+\underbrace{2\tau^2\alpha[2ac+(2+2a)-\alpha]}_{O(\tau^2)} \\
(A+\zeta)\ell_x =& \tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-(d_x^2+ad_y^2)\right]d_x+\tau^2[2ac+(d_{xx}+ad_{yy})-\alpha]d_x+\tau^2\alpha d_x \\
=& \tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]d_x+\tau^2[2ac+(2+2a)]d_x \\
\frac{\partial}{\partial x}[(A+\zeta)\ell_x] =& \tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-(d_x^2+ad_y^2)\right]d_{xx}-\tau^3[2d_xd_{xx}+2ad_yd_{yx}]d_x+O(\tau^2) \\
=& 2\tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]-16\tau^3(x-x_0)^2+O(\tau^2) \\
(A+\zeta)\ell_y =& \tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]d_y+\tau^2[2ac+(2+2a)]d_y \\
\frac{\partial}{\partial y}[(A+\zeta)\ell_y] =& 2\tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]-16a\tau^3(y-y_0)^2+O(\tau^2) \\
\frac{\partial}{\partial t}[(A+\zeta)\ell_t)] =& 2ac\tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-(d_x^2+ad_y^2)\right]+2ac\tau^3\left(t-\frac{T}{2}\right)^2 8a^2c^2 \\
=& 2ac\tau^3\left[4a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]+16a^3c^3\left(t-\frac{T}{2}\right)^2 \\
=& 2ac\tau^3\left[12a^2c^2\left(t-\frac{T}{2}\right)^2-4\left((x-x_0)^2+a(y-y_0)^2\right)\right]
\end{aligned}
$$

Thus, making the appropriate substitutions into the defined $\tilde{B}$ at the beginning of the section yields the result in (3.7). Morever, applying the specializations in (3.2), (3.3), and (3.4) to the estimate in (3.1) changes the estimate to

$$
\begin{aligned}
&\theta^2[\psi_{tt}-(\psi_{xx}+a\psi_{yy})]^2-\frac{\partial}{\partial t}M+\frac{\partial}{\partial x}V_1+a\frac{\partial}{\partial y}V_2 \\
&\geq 2\left[2\tau(1+a)-2ac\tau-\tau\alpha\right]v_t^2+2\left[\tau\alpha-\frac{\epsilon}{2}-2\tau(1+a)-2ac\tau\right](v_x^2+av_y^2)+4(2\tau v_x^2+a^2 2\tau v_y^2)+\theta^2\tilde{B}\psi^2
\end{aligned}
$$

which simplifies to the estimate in (3.5).

□

Since Theorem 3.2.1 holds for an arbitrary constant α, let

$$\alpha \equiv (2+2a) - 2ac - a(1-k) \quad \text{for } 0 < k < 1$$

such that $(2+2a) - 2ac - \alpha = a(1-k) > 0$. Moreover, if we define γ by

$$\gamma \equiv \alpha - 2ac - (2+2a) = -4ac - a(1-k) < 0$$

then we can choose a positive constant ρ by

$$\rho \equiv 4a + \gamma = 4a - 4ac - a(1-k) > 0 \quad \text{for } 4c - 3 < k < 1$$

and the inequality $(2+2a) - 2ac - \alpha \geq \rho > 0$ also holds.

Remark: *While we have set ρ equivalent to $4a+\gamma$, it is possible to set ρ slightly less than $4a+\gamma$ for the purposes of the following corollary.*

Thus we have the necessary conditions to establish the following pointwise inequality:

Corollary 3.2.1.1. *Following the aforementioned specifications for our choice of α, γ, and ρ we have the following improvement on the estimate in Thoerem 3.2.1 as*

$$\theta^2 \left[\psi_{tt} - (\psi_{xx} + a\psi_{yy})\right]^2 - \frac{\partial}{\partial t}M + \frac{\partial}{\partial x}V_1 + a\frac{\partial}{\partial y}V_2 \geq 2\tau\rho[v_t^2 + (v_x^2 + av_y^2)] + \tilde{B}v^2. \tag{3.8}$$

for all $0 \leq t \leq T$ and $(x,y) \in \Omega$. Additionally, for a positive constant $\tilde{\beta}$, we have the estimate

$$\tilde{B}v^2 \geq [2\tau^3\tilde{\beta} + O(\tau^2)]v^2 \text{ for all } (x,y,t) \in Q(\sigma) = \{(x,y,t)|P(x,y,t) \geq \sigma > 0\} \tag{3.9}$$

where $0 < \sigma < \min_{(x,y)\in\bar{\Omega}} \left((x-x_0)^2 + (y-y_0)^2\right)$ as established in the beginning of the chapter.

Proof. With the assumptions on ρ, γ and α, and recalling $v = \theta\psi$ we have the following bound on the right hand side of the inequality in (3.5)

$$2\tau\left[(2+2a) - 2ac - \alpha\right]v_t^2 + 2\tau\left[\alpha - \frac{\epsilon}{2\tau} - (2+2a) - 2ac\right](v_x^2 + av_y^2) + 4\tau(2v_x^2 + a^2 2v_y^2) + \theta^2\tilde{B}\psi^2$$
$$\geq 2\tau\rho v_t^2 + 2\tau\left[\alpha - \frac{\epsilon}{2\tau} - (2+2a) - 2ac + 4a\right](v_x^2 + av_y^2) + \tilde{B}v^2.$$

This can be taken further by simplifying the coefficient of the $(v_x^2+v_y^2)$ term as follows

$$\alpha-\frac{\epsilon}{2\tau}-(2+2a)-2ac+4a=\gamma+4a-\frac{\epsilon}{2\tau}\geq\rho$$

by the arbitrary nature of ϵ. Thus we have

$$2\tau\left[(2+2a)-2ac-\alpha\right]v_t^2+2\tau\left[\alpha-\frac{\epsilon}{2\tau}-(2+2a)-2ac\right](v_x^2+av_y^2)+4\tau(2v_x^2+a^22v_y^2)+\theta^2\tilde{B}\psi^2$$
$$\geq 2\tau\rho[v_t^2+(v_x^2+av_y^2)]+\tilde{B}v^2$$

as desired. For the part of the corollary regarding the bound on $\tilde{B}$ notice from how α was defined and the assumptions we have made for the function P, we have for all $(x,y,t)\in Q(\sigma)$

$$\begin{aligned}\tilde{B}&=2\tau^3\left\{4[2ac+(2+2a)-\alpha][(x-x_0)^2+a(y-y_0)^2]+16(x-x_0)^2+16a^2(y-y_0)^2\right.\\&\left.-[6ac+(2+2a)-\alpha]4a^2c^2\left(t-\frac{T}{2}\right)^2\right\}+O(\tau^2)\\&\geq 2\tau^3\left\{4[4ac+a(1-k)]a[(x-x_0)^2+(y-y_0)^2]+16a^2[(x-x_0)^2+(y-y_0)^2]\right.\\&\left.-[8ac+a(1-k)]4a^2c^2\left(t-\frac{T}{2}\right)^2\right\}+O(\tau^2).\end{aligned}$$

Hence, for all $(x,y,t)\in Q(\sigma)$ we have

$$4a[(4ac+a(1-k))+4a][(x-x_0)^2+(y-y_0)^2]-[8ac+a(1-k)]4a^2c^2\left(t-\frac{T}{2}\right)^2>0$$

Thus there exists a positive constant $\tilde{\beta}$ for the given domain satisfying (3.9).

□

In (3.8) there is still the issue of the left-hand side being expressed in terms of ψ while the right-hand side is in terms of v. The following corollary provides an estimate that corrects this issue.

Corollary 3.2.1.2. *For an arbitrary ϵ (different than the previous ϵ used) such that $1>\epsilon>0$,* *from Corollary 3.2.1.1 we obtain*

$$\theta^2\left[\psi_{tt}-(\psi_{xx}+a\psi_{yy})\right]^2-\frac{\partial}{\partial t}M+\frac{\partial}{\partial x}V_1+a\frac{\partial}{\partial y}V_2\geq\epsilon\tau\rho\theta^2[\psi_t^2+(\psi_x^2+a\psi_y^2)]+[\tilde{B}-2\epsilon\rho\tau^3r]\theta^2\psi^2 \tag{3.10}$$

for $0\leq t\leq T,(x,y)\in\Omega$ *and where* $r=\max\left\{P_t^2+P_x^2+aP_y^2\right\}$.

Proof. Recall $\ell_t=\tau P_t,\ell_x=\tau P_x,\ell_y=\tau P_y$, thus we have $v_t=\theta(\psi_t+\ell_t\psi)=\theta\psi_t+\theta\tau P_t\psi$ and similarly $v_x=\theta(\psi_x+\ell_x\psi)=\theta\psi_x+\theta\tau P_x\psi$, $v_y=\theta(\psi_y+\ell_y\psi)=\theta\psi_y+\theta\tau P_y\psi$. So $\theta\psi_t=v_t-\theta\tau P_t\psi$ and hence

$$\theta^2\psi_t^2\leq 2v_t^2+2\theta^2\tau^2P_t^2\psi^2\Rightarrow 2v_t^2\geq\theta^2\psi_t^2-2\theta^2\tau^2P_t^2\psi^2.$$

Similarly, we have

$$2v_x^2\geq\theta^2\psi_x^2-2\theta^2\tau^2P_x^2\psi^2;\ 2v_y^2\geq\theta^2\psi_y^2-2\theta^2\tau^2P_y^2\psi^2.$$

Applying these results to the right-hand side of (3.8), and utilizing the defined ϵ yields

$$\begin{aligned}2\tau\rho[v_t^2+(v_x^2+av_y^2)]+\tilde{B}v^2&\geq\epsilon\tau\rho[2v_t^2+(2v_x^2+2av_y^2)]+\tilde{B}v^2\\&\geq\epsilon\tau\rho\theta^2[\psi_t^2+(\psi_x^2+a\psi_y^2)]+\tilde{B}\theta^2\psi^2-2\epsilon\tau^3\rho\theta^2[P_t^2+P_x^2+aP_y^2]\psi^2\\&\geq\epsilon\tau\rho\theta^2[\psi_t^2+(\psi_x^2+a\psi_y^2)]+[\tilde{B}-2\epsilon\rho\tau^3r]\theta^2\psi^2\end{aligned}$$

which gives the desired result. □

For the purposes of establishing the Carleman estimate in the next section we shall rewrite (3.10) with the following definition

$$\theta^2\left[\psi_{tt}-(\psi_{xx}+a\psi_{yy})\right]^2-\frac{\partial}{\partial t}M+\frac{\partial}{\partial x}V_1+a\frac{\partial}{\partial y}V_2\geq\epsilon\tau\rho\theta^2[\psi_t^2+(\psi_x^2+a\psi_y^2)]+B\theta^2\psi^2 \tag{3.11}$$

where $B\equiv\tilde{B}-2\epsilon\rho\tau^3[P_t^2+(P_x^2+aP_y^2)]\geq\tilde{B}-2\epsilon\rho\tau^3r$. Thus making ϵ sufficiently small gives

$$B\psi^2\geq[2\tau^2\beta+O(\tau^2)]\psi^2,\forall(x,y,t)\in Q(\sigma)$$

where β is a positive constant whose dependence on ϵ is defined by $\beta=(\tilde{\beta}-\epsilon\rho r)>0$ and $B=O(\tau^3)$ in $[0,T]\times\Omega$.

3.3 Carleman estimate for equations (1.3) and (1.4)

Theorem 3.3.1. *With the established assumptions from section 3.1 as well as well as resulting conclusions in the succeeding sections, then the following estimates hold for any small* $\epsilon > 0$ *and* $\tau > 0$ *sufficiently large:*

$$\begin{aligned} & BT|_{\Sigma}^{\psi} + C\int_0^T \int_{\Omega} e^{2\tau P}(\phi_{xy}^2 + w_x^2)dxdydt + C_{1,T}e^{2\tau\sigma}\int_{[Q(\sigma)]^c} \psi^2 dxdydt \\ & \geq (\epsilon\tau\rho - 2C)\int_0^T \int_{\Omega} e^{2\tau P}[\psi_t^2 + \psi_x^2 + a\psi_y^2]dxdydt \\ & + (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt - C_T\tau^3 e^{-2\tau\delta}[E_\psi(0) + E_\psi(T)] \end{aligned} \tag{3.12}$$

where $E_\psi(t)$ *and* $BT|_{\Sigma}^{\psi}$ *are as defined in* (3.14) *and* (3.15) *respectively.*

Proof. The initial step is to integrate (3.11) over $Q \equiv [0,T] \times \Omega$ where we separate Q as $Q = Q(\sigma) \cup [Q(\sigma)]^c$ since (3.9) holds necessarily on $Q(\sigma)$. This yields

$$\begin{aligned} & \int_0^T \int_{\Omega} \theta^2 \left[\psi_{tt} - (\psi_{xx} + a\psi_{yy})\right]^2 dxdydt - \left[\int_{\Omega} \frac{\partial}{\partial t} M\right]_0^T + \int_0^T \int_{\Omega} \frac{\partial}{\partial x} V_1 + a\frac{\partial}{\partial y} V_2 dxdydt \\ & \geq \epsilon\tau\rho \int_0^T \int_{\Omega} \theta^2[\psi_t^2 + (\psi_x^2 + a\psi_y^2)]dxdydt + \int_0^T \int_{\Omega} B\theta^2\psi^2 dxdydt. \end{aligned} \tag{3.13}$$

We will evaluate the terms separately, recalling pertinent details as they become relevant. Beginning with the right-hand side, notice the estimate for B only holds true on $Q(\sigma)$ thus we have

$$\begin{aligned} \int_0^T \int_{\Omega} \theta^2 B\psi^2 dxdydt &= \int_{Q(\sigma)} \theta^2 B\psi^2 dxdydt + \int_{[Q(\sigma)]^c} \theta^2 B\psi^2 dxdydt \\ &\geq [2\tau^2\beta + O(\tau^2)]\int_{Q(\sigma)} \theta^2 dxdydt + \int_{[Q(\sigma)]^c} \theta^2 B\psi^2 dxdydt. \end{aligned}$$

Focusing on the left-hand side, we have, using a trivial inequality resulting from (1.3), the result

$$\begin{aligned} & \int_0^T \int_{\Omega} \theta^2 \left[\psi_{tt} - (\psi_{xx} + a\psi_{yy})\right]^2 dxdydt \\ & \leq 2C\left[\int_0^T \int_{\Omega} \theta^2\psi^2 dxdydt\right] + 2C_a\int_0^T \int_{\Omega} \theta^2(\phi_{xy}^2 + w_x^2)dxdydt. \end{aligned}$$

Next, recalling

$$M = \theta^2[-2\ell_t(\psi_t^2 + \psi_x^2 + a\psi_y^2) + 4\psi_t(\ell_x\psi_x + a\ell_y\psi_y) + 2(2\ell_x^2 + 2a\ell_y^2 - 2\ell_t^2 + \zeta)\psi\psi_t$$
$$+ (2\ell_t(\ell_x^2 + a\ell_y^2) - 2\ell_t^3 - 2A\ell_t - \zeta_t)\psi^2]$$

we have, via the specializations in section 3.1,

$$M = \theta^2 \left[4ac\tau \left(t - \frac{T}{2}\right) (\psi_t^2 + \psi_x^2 + a\psi_y^2) + 8\tau\psi_t \left[(x - x_0)\psi_x + a(y - y_0)\psi_y\right] \right.$$
$$+ 2\left[4\tau^2(x - x_0)^2 + 8a\tau^2(y - y_0)^2 - 8a^2c^2\tau^2 \left(t - \frac{T}{2}\right)^2 + \tau(2 + 2a - 2ac - a(1 - k)\right]\psi\psi_t$$
$$+ \left(-4ac\tau \left(t - \frac{T}{2}\right) 4\tau^2 \left[(x - x_0)^2 + a(y - y_0)^2\right] + 16a^3c^3\tau^3 \left(t - \frac{T}{2}\right)^3 \right.$$
$$\left. \left. +8ac\tau^2 \left(4a^2c^2 \left(t - \frac{T}{2}\right)^2 - \left[4(x - x_0)^2 + 4a(y - y_0)^2\right] + \tau(2 + 2a + 2ac - \alpha)\right) \left(t - \frac{T}{2}\right)\right)\psi^2 \right].$$

Thus, by the highest order on τ we observe

$$M \leq C_{T,a}\tau^3\theta^2[\psi_t^2 + \psi_x^2 + a\psi_y^2 + \psi^2]$$

and by the Poincare inequality then

$$M \leq C_{T,a}\tau^3\theta^2[\psi_t^2 + \psi_x^2 + \psi_y^2].$$

Defining the energy term as

$$E_\psi(t) \equiv \frac{1}{2}\int_\Omega [\psi_t^2 + \psi_x^2 + \psi_y^2]dxdy \tag{3.14}$$

the integral term of M on the left hand side of (3.13) then has the following estimate

$$\begin{aligned}\left[\int_{\Omega} M dxdy\right]_0^T &\leq C_T\tau^3\left[\int_{\Omega} e^{2\tau P}[\psi_t^2+\psi_x^2+\psi_y^2]\right]\\ &\leq C_T\tau^3 e^{-2\tau\delta}\left[\int_{\Omega}[\psi_t^2+\psi_x^2+\psi_y^2]\right]\\ &\leq C_T\tau^3 e^{-2\tau\delta}\left[E_\psi(0)+E_\psi(T)\right]\end{aligned}$$

where we drop the dependence of the constant on a since a is bounded as defined previously. Additionally, notice the middle inequality arises as a result of properties of P defined in section 3.1. For the next integral we have

$$\int_0^T\int_{\Omega}\frac{\partial}{\partial x}V_1+a\frac{\partial}{\partial y}V_2dxdydt=\int_0^T\int_{\Omega}\operatorname{div}(V_1,aV_2)dxdydt=\int_0^T\int_{\partial\Omega}(V_1,aV_2)\cdot(\nu_1,\nu_2)dSdt.$$

To analyze the dot product in the final equality statement, first we rewrite V_1 and V_2 via the specializations in section 3.1. This gives

$$\begin{aligned}V_1 =&2\theta^2\left(2\psi_x\left[2\tau(x-x_0)\psi_x+2a\tau(y-y_0)\psi_y+2ac\tau\left(t-\frac{T}{2}\right)\psi_t\right]-2\tau(x-x_0)(\psi_x^2+a\psi_y^2-\psi_t^2)\right.\\ &+2\left(4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2+\frac{\zeta}{2}\right)\psi_x\psi\\ &\left.2\tau(x-x_0)\left(4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2-A\right)\psi^2\right)\\ aV_2 =&2a\theta^2\left(2\psi_y\left[2\tau(x-x_0)\psi_x+2a\tau(y-y_0)\psi_y+2ac\tau\left(t-\frac{T}{2}\right)\psi_t\right]-2\tau(y-y_0)(\psi_x^2+a\psi_y^2-\psi_t^2)\right.\\ &+2\left(4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2+\frac{\zeta}{2}\right)\psi_y\psi\\ &\left.2\tau(y-y_0)\left(4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2-A\right)\psi^2\right)\end{aligned}$$

where we can simplify the following expression using the same specializations of partial derivatives of ℓ in the definition of A as

$$
\begin{aligned}
&4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2-A\\
=&8\tau^2(x-x_0)^2+8a\tau^2(y-y_0)^2-8a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2+\tau(\alpha-2ac-2a-2).
\end{aligned}
$$

For the purposes of easing the notation, we will maintain the use of the following definition of an operator for the rest of the paper:

$$\nabla_a^\psi f \equiv (f_x, af_y) \text{ for some } f\in C^1(\mathbb{R}^2).$$

Then, with $\nu=(\nu_1,\nu_2)$, the inner product $(V_1,aV_2)\cdot(\nu_1,\nu_2)=V_1\nu_1+aV_2\nu_2$ can be written in terms of the above expressions for V_1 and V_2 as

$$
\begin{aligned}
&2\theta^2\left\{\tau[\psi_t^2-(\psi_x^2+a\psi_y^2)]\nabla_a d\cdot\nu 4ac\tau\left(t-\frac{T}{2}\right)\psi_t\nabla_a\psi\cdot\nu+2[2\tau(x-x_0)\psi_x+2a\tau(y-y_0)\psi_y]\nabla_a\psi\cdot\nu\right.\\
&\quad+2\left[4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2+\frac{\zeta}{2}\right]\psi\nabla_a\psi\cdot\nu\\
&\quad\left.+\tau\left[4\tau^2(x-x_0)^2+4a\tau^2(y-y_0)^2-4a^2c^2\tau^2\left(t-\frac{T}{2}\right)^2-A\right]\psi^2\nabla_a d\cdot\nu\right\}.
\end{aligned}
$$

So we can write the boundary terms with $\Sigma=[0,T]\times\partial\Omega$ in the following expanded form:

$$
\begin{aligned}
&BT|_\Sigma^\psi=\int_0^T\int_{\partial\Omega}(V_1,aV_2)\cdot(\nu_1,\nu_2)dSdt\\
&=2\tau\int_0^T\int_{\partial\Omega}e^{2\tau P}[\psi_t^2-(\psi_x^2+a\psi_y^2)]\nabla_a^\psi d\cdot\nu dSdt\\
&+8ac\tau\int_0^T\int_{\partial\Omega}e^{2\tau P}\left(t-\frac{T}{2}\right)\psi_t\nabla_a^\psi\psi\cdot\nu dSdt+8\tau\int_0^T\int_{\partial\Omega}e^{2\tau P}[\nabla d\cdot\nabla_a^\psi\psi]\nabla_a^\psi\psi\cdot\nu dSdt\\
&+4\tau^2\int_0^T\int_{\partial\Omega}e^{2\tau P}\left[4(x-x_0)^2+4a(y-y_0)^2-4a^2c^2\left(t-\frac{T}{2}\right)^2+\frac{\alpha\tau}{2}\right]\psi\nabla_a^\psi\psi\cdot\nu dSdt\\
&+2\tau\int_0^T\int_{\partial\Omega}e^{2\tau P}\left\{8\tau^2\left[(x-x_0)^2+a(y-y_0)^2-a^2c^2\left(t-\frac{T}{2}\right)^2\right]+\tau(\alpha-2ac-2a-2)\right\}\psi^2\nabla_a^\psi d\cdot\nu dSdt.
\end{aligned}
\tag{3.15}
$$

Applying these results to (3.13) produces

$$
\begin{aligned}
&BT|_{\Sigma}^{\psi} + C_a \int_0^T \int_\Omega e^{2\tau P}(\phi_{xy}^2 + w_x^2)dxdydt - \int_{[Q(\sigma)]^c} e^{2\tau P} B\psi^2 dxdydt \\
&\quad \geq (\epsilon\tau\rho - 2C)\int_0^T \int_\Omega e^{2\tau P}[\psi_t^2 + \psi_x^2 + a\psi_y^2]dxdydt + (2\tau^3\beta + O(\tau^2))\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt \\
&\quad - 2C\int_0^T \int_\Omega e^{2\tau P}\psi^2 dxdydt - C_T\tau^3 e^{-2\tau\delta}[E_\psi(0) + E_\psi(T)].
\end{aligned} \tag{3.16}
$$

Moreover, recall $B = O(\tau^3)$ in $[0, T] \times \Omega$ and $P(x, y, t) \leq \sigma$ on $[Q(\sigma)]^c$. The term involving B on the left-hand side of (3.16) hence has the estimate

$$
-\int_{[Q(\sigma)]^c} e^{2\tau P} B\psi^2 dxdydt \leq e^{2\tau\sigma} O(\tau^3)\int_{[Q(\sigma)]^c} \psi^2 dxdydt.
$$

Examining the right-hand side of the inequality (3.16), we have

$$
\int_0^T e^{2\tau P}\psi^2 dxdydt = \int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt + \int_{[Q(\sigma)]^c} e^{2\tau P}\psi^2 dxdydt.
$$

Hence,

$$
\begin{aligned}
&(2\tau^3\beta + O(\tau^2))\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt - 2C\int_0^T \int_\Omega e^{2\tau P}\psi^2 dxdydt \\
&= (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt - 2C\int_{[Q(\sigma)]^c} e^{2\tau P}\psi^2 dxdydt \\
&\geq (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt - 2Ce^{2\tau\sigma}\int_{[Q(\sigma)]^c} \psi^2 dxdydt.
\end{aligned}
$$

Substituting these results into (3.16) and manipulating the terms gives

$$
\begin{aligned}
&BT|_{\Sigma}^{\psi} + C_a \int_0^T \int_\Omega e^{2\tau P}(\phi_{xy}^2 + w_x^2)dxdydt + (O(\tau^3)e^{2\tau\sigma} + 2Ce^{2\tau\sigma})\int_{[Q(\sigma)]^c} \psi^2 dxdydt \\
&\geq (\epsilon\tau\rho - 2C)\int_0^T \int_\Omega e^{2\tau P}[\psi_t^2 + \psi_x^2 + a\psi_y^2]dxdydt \\
&+ (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\sigma)} e^{2\tau P}\psi^2 dxdydt - C_T\tau^3 e^{-2\tau\delta}[E_\psi(0) + E_\psi(T)].
\end{aligned}
$$

Thus, defining the constant $C_{1,T}e^{2\tau\sigma} = O(\tau^3)e^{2\tau\sigma} + 2Ce^{2\tau\sigma}$ produces the final result. □

Similarly, we have the Carleman estimate for equation (1.4) below

Theorem 3.3.2. *Again, with the established assumptions from theorem 3.3.1, then the following estimates hold for any small $\epsilon > 0$ and $\tau > 0$ sufficiently large:*

$$\begin{aligned} BT|_{\Sigma}^{\phi} &+ C\int_0^T\int_{\Omega} e^{2\tau P}(\psi_{xy}^2 + w_y^2)dxdydt + C_{1,T}e^{2\tau\sigma}\int_{[Q(\sigma)]^c}\phi^2 dxdydt \\ &\geq (\epsilon\tau\rho - 2C)\int_0^T\int_{\Omega} e^{2\tau P}[\phi_t^2 + a\phi_x^2 + \phi_y^2]dxdydt \\ &+ (2\tau^3\beta + O(\tau^2) - 2C)\int_{Q(\sigma)} e^{2\tau P}\phi^2 dxdydt - C_T\tau^3 e^{-2\tau\delta}[E_{\phi}(0) + E_{\phi}(T)]. \end{aligned} \tag{3.17}$$

with $E_{\phi}(t)$ defined as

$$E_{\phi}(t) \equiv \frac{1}{2}\int_{\Omega}[\phi_t^2 + \phi_x^2 + \phi_y^2]dxdy$$

and where, using the notation $\nabla_a^{\phi} f = (af_x, f_y)$ (the weight appears in the first component), we have

$$\begin{aligned} BT|_{\Sigma}^{\phi} &= 2\tau\int_0^T\int_{\partial\Omega} e^{2\tau P}[\phi_t^2 - (a\phi_x^2 + \phi_y^2)]\nabla_a^{\phi} d\cdot\nu dSdt \\ &+ 8ac\tau\int_0^T\int_{\partial\Omega} e^{2\tau P}\left(t - \frac{T}{2}\right)\phi_t\ \nabla_a^{\phi}\phi\cdot\nu dSdt + 8\tau\int_0^T\int_{\partial\Omega} e^{2\tau P}[\nabla d\cdot\nabla_a^{\phi}\phi]\nabla_a^{\phi}\phi\cdot\nu dSdt \\ &+ 4\tau^2\int_0^T\int_{\partial\Omega} e^{2\tau P}\left[4a(x-x_0)^2 + 4(y-y_0)^2 - 4a^2c^2\left(t-\frac{T}{2}\right)^2 + \frac{\alpha\tau}{2}\right]\phi\nabla_a^{\phi}\phi\cdot\nu dSdt \\ &+ 2\tau\int_0^T\int_{\partial\Omega} e^{2\tau P}\left\{8\tau^2\left[a(x-x_0)^2 + (y-y_0)^2 - a^2c^2\left(t-\frac{T}{2}\right)^2\right] + \tau(\alpha - 2ac - 2a - 2)\right\}\phi^2\nabla_a^{\phi} d\cdot\nu dSdt. \end{aligned}$$

3.4 Carleman estimate for the Mindlin-Timoshenko model

Taking the sum of the estimates for equations (1.2), (1.3), and (1.4) produces a one parameter family of estimates for the Mindlin-Timoshenko model in its totality. First, let $E(t)$ and $BT|\Sigma$ be given by

$$E(t) \equiv E_w(t) + E_{\psi}(t) + E_{\phi}(t)$$
$$BT|_{\Sigma} \equiv BT|_{\Sigma}^{w} + BT|_{\Sigma}^{\psi} + BT|_{\Sigma}^{\phi}.$$

Thus adding (1.5), (3.12), and (3.17) by choosing the maximum value of C from the three estimates since these values may vary, the maximum value of $C_{1,t}$, the maximum C_T, the maximum δ and the minimum ρ and β produces

$$
\begin{aligned}
&BT|_\Sigma + C\int_0^T \int_\Omega e^{2\tau P}(\psi_{xy}^2 + \phi_{xy}^2 + |\nabla w|^2) + e^{2\tau \tilde{P}}(\psi_x + \phi_y)^2 dxdydt \\
&+ C_{1,T}\left[e^{2\tau\sigma}\int_{[Q(\sigma)]^c} \psi^2 + \phi^2 dxdydt + e^{2\tau\tilde{\sigma}}\int_{[Q(\tilde{\sigma})]^c} w^2 dxdydt\right] \\
&\geq (\epsilon\tau\rho - 2C)\left[\int_0^T \int_\Omega e^{2\tau P}(\psi_t^2 + \phi_t^2 + a\psi_x^2 + \phi_x^2 + \psi_y^2 + \phi_y^2)dxdydt + \int_0^T \int_\Omega e^{2\tau\tilde{P}}(w_t^2 + |\nabla w|^2)dxdydt\right] \\
&+ (2\tau^3\beta + O(\tau^2) - 2C)\left[\int_{Q(\sigma)} e^{2\tau P}(\psi^2 + \phi^2)dxdydt + \int_{Q(\tilde{\sigma})} e^{2\tau\tilde{P}} w^2 dxdydt\right] - C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)].
\end{aligned}
$$

Chapter 4

Conclusions and Discussion

The overall result achieved by the research presented was the final carleman estimate for the Mindlin-Timoshenko model, with boundary terms, given by

$$
\begin{aligned}
&BT|_{\Sigma} + C\int_0^T \int_{\Omega} e^{2\tau P}(\psi_{xy}^2 + \phi_{xy}^2 + |\nabla w|^2) + e^{2\tau \tilde{P}}(\psi_x + \phi_y)^2 dxdydt \\
&+ C_{1,T}\left[e^{2\tau\sigma}\int_{[Q(\sigma)]^c} \psi^2 + \phi^2 dxdydt + e^{2\tau\tilde{\sigma}}\int_{[Q(\tilde{\sigma})]^c} w^2 dxdydt\right] \\
&\geq (\epsilon\tau\rho - 2C)\left[\int_0^T \int_{\Omega} e^{2\tau P}(\psi_t^2 + \phi_t^2 + a\psi_x^2 + \phi_x^2 + \psi_y^2 + \phi_y^2)dxdydt + \int_0^T \int_{\Omega} e^{2\tau\tilde{P}}(w_t^2 + |\nabla w|^2)dxdydt\right] \\
&+ (2\tau^3\beta + O(\tau^2) - 2C)\left[\int_{Q(\sigma)} e^{2\tau P}(\psi^2 + \phi^2)dxdydt + \int_{Q(\tilde{\sigma})} e^{2\tau\tilde{P}} w^2 dxdydt\right] - C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)].
\end{aligned}
$$

which has not previously been published. Notice, for sufficiently large τ, the terms

$$
\begin{aligned}
&C_{1,T}\left[e^{2\tau\sigma}\int_{[Q(\sigma)]^c} \psi^2 + \phi^2 dxdydt + e^{2\tau\tilde{\sigma}}\int_{[Q(\tilde{\sigma})]^c} w^2 dxdydt\right]; \\
&- C_T\tau^3 e^{-2\tau\delta}[E(0) + E(T)]
\end{aligned}
$$

will either vanish or be absorbed in the estimate and hence are included for detail, but not apart of the general estimate. The general Carleman estimate simply includes the principal part of the model with boundary terms on the left-hand side, while the right-hand side consists of lower level energy terms.

4.1 Purpose of the Estimate

The original purpose of Carleman estimates were mostly to prove unique continuation theorems, but this has evolved over time. The inclusion of the exact boundary terms make this estimate useful for boundary control problems in the field of control theory. Observability for the model would also need to be established since the model is different enough from the traditional wave equation that this would not be guaranteed. Another application would be utilizing the estimate in exploring the inverse problem for this model.

4.2 Recommendations for Further Research

The next step will be to pursue observability and, hence, exact controllability of the Mindlin-Timoshenko model, which would open the possibility of a myriad of applications for this model. Due to the nature of the model's application to the mechanics of vibrating, thin plates, engineers and applied mathematicians who work with such models could use the information to further there own research. These systems of thin plates under high frequency vibrations have appeared in proximity sensors and other electronic devices so the implications are broad [10].

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