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The Effects of Viscoelastic Behavior on the Operation of a Delayed Resonator Vibration Absorber

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THE EFFECTS OF VISCOELASTIC BEHAVIOR ON THE OPERATION OF A DELAYED RESONATOR VIBRATION ABSORBER

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mechanical Engineering

by
John Quentin Cowans
December 2006

Accepted by:
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ABSTRACT

Delayed resonators have proven to be effective vibration absorbers (VAs) for tracking and canceling the effects of harmonic excitations on a structure. The Delayed Resonator (DR) is self-contained, as no information from outside of its substructure is required for proper operation. It adjusts for variations in frequency using time-delay and gain as control parameters. Proper calculation of these control parameters is dependent upon accurate knowledge of the absorber’s structural properties. Stability is of great concern when the DR is used because it operates in a state of marginal stability. Determining stability limits accurately is dependent upon a correct assessment of the system properties such as stiffness and coefficient of damping.

This thesis examines the relationship between viscoelastic (VE) loss mechanisms in systems with DR and the choice of modeling method used to calculate control parameters and determine system stability. It is hypothesized that a VE loss mechanism approximated by a single viscous dashpot may lead to unexpected limits on the DR’s performance and adversely effect system stability. The constitutive properties of viscoelastic materials are dependent on both time and temperature, while the idealized viscous damper’s damping coefficient is not affected by either. The response of strongly VE structures to excitations may deviate widely from that which can be predicted by a viscous model. Maxwell Standard Models (MSMs) are used to simulate the time-dependence in the system, and temperature dependence is included via the time-temperature superposition principle.

The hypothesis is tested by simulating three possible combinations of host and/or DR modeled with VE and viscous loss mechanisms (i.e., host-viscous and DR-VE; host-VE and DR-viscous; host-VE and DR-VE7). Results show a modest level of performance enhancement when MSMs are used in place of viscous models when calculating control parameters. If the viscous loss mechanism is used to calculate the control parameters when the DR substructure has a VE loss mechanism, the oscillations in the host are damped, but not to the same degree as when the VE model is used in the calculation. However, the assessment of stability of the system is greatly improved when a MSM is used over a viscous model in the determination of the stability limits. In some cases, the viscous models of systems with VE damping predict that the system is only stable above a certain frequency, while the MSM predict that the system is only stable between two frequencies, which can vary with temperature. We expect that these findings would be most...
useful in the design of DR systems where the VE damping mechanism plays a significant role either purposefully or due to inherent structural properties.
DEDICATION

This document is dedicated to my family and friends. Especially appreciate my mother for providing me with encouragement as well as food and shelter during the my time at Clemson.
ACKNOWLEDGMENTS

I would like to thank Dr. Austin for helping me and providing guidance over the years. I would like to thank Dr. Haque and Dr. Jalili for participating on my committee and helping me complete this process successfully.
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CHAPTER 1

Introduction

1.1 Background

The study of structural vibrations has fascinated the engineering community for many years. However, structural vibrations are not just a concern of engineers. They are commonplace in daily life. Vibrations can be desirable and pleasant; think of the gentle sound from the vibrating strings of a cello. Unfortunately, vibrations are often unacceptable, such as a skyscraper swaying in the wind. This swaying presents many problems; it compromises the structural integrity of the building through fatigue. The swaying can, also, cause the buildings occupants to become sick.

In many cases, the vibrations observed in engineering structures are a result of harmonic forces acting on the structure, i.e., forces whose magnitude repeats in time at a given frequency. Automobile engines impart harmonic excitations on automotive frames resulting in vibrations that can lead to unacceptable levels of noise. It is common for airplane wings experiencing turbulent winds to oscillate during flight causing mechanical fatigue. Often times controlling and suppressing unwanted vibrations reduces to negating the effects of harmonic forces. Today’s engineers employ various techniques to prevent or reduce the response of structures to harmonic excitations. One of the most common and oldest methods is the use of the auxiliary mass vibration absorber. It was invented by Frahm (1911) in the introduction to his United States patent, he stated,

This invention relates to a means for damping the resonance-vibrations which arise in bodies subjected to certain periodic impacts. Such bodies are for instance ships which are subjected to periodic vibratory forces from their propelling machinery or from their propellers. As is known in such ships as soon as the impacts are in harmony with the natural oscillations of the ship the ship starts vibrating more or less. Such vibrations however are not confined to ships, but are also present in airships, aeroplanes and railway and street vehicles. Further the same phenomenon are evident in fixed bodies such as buildings when vehicles pass near them, or when machines are working within them.

Frahm’s invention is often called a tuned-mass vibration absorber (TMVA): a single-degree-of-freedom (SDOF) oscillator attached to the excited or host system.

Theoretically, a TMVA can completely cancel the effect of harmonic excitations of any size by splitting a resonance of the host system into two and placing a zero at the tuned frequency. Figure
Figure 1.1 Shows example schematics for a TMVA and a TMD.

1.1 shows schematics for the TMVA and the TMD while Fig. 1.2 compares the frequency response of a SDOF undamped oscillator with and without the above vibration suppression treatments applied. Unfortunately, a TMVA is only useful when the excitation frequency varies over a narrow band. Variations in the excitation frequency may inadvertently excite one of the neighboring resonances. The tuned-mass damper (TMD) is a variation of Frahm’s invention where a viscous damper is added in parallel with the spring of the SDOF oscillator. Although it does not have the theoretical ability to cancel the effects of the excitations completely, it is useful in attenuating the response of a structure over a wider range of frequencies.

Recently developed auxiliary mass vibration cancelation devices combine the best characteristics of the TMVA and the TMD; they completely cancel vibrations over a broad range of frequencies. Unlike their predecessors, many modern devices are semiactive, i.e., they rely on sensors and electronics to detect and control vibrations. Figure 1.3, redrawn from Kwok and Samali (1995), shows a simplified representation of a modern device attached to a skyscraper. The device “tunes” itself in real time through use of computer algorithms. The sensor provides position or its derivative information about the system to the controller. The computer uses sensor inputs to devise a control signal that is sent to the actuator to minimize the effect of the excitation on the host structure. A recent improvement on these devices is the delayed resonator, an active tuned-mass vibration damper developed by Olgac and Holm-Hansen (1994). The foremost improvement is that
Figure 1.2  FRF’s of SDOF oscillator with no vibration absorption treatment, with TMVA, and with TMD.

Figure 1.3  Single-degree-of-freedom wind-loaded system fitted with active TMD.
this device is self contained. It needs only its own position or derivative data for tuning, no host system parameters are needed.

An alternative method of vibration control is to damp the system response rather than cancel the input. In structures this is typically done through damping treatments that do not require an auxiliary mass. Engineers attach viscoelastic (VE) dampers, VE materials including rubber and polymers, via lamination or through the replacement of mechanical joints. The VE dampers dissipate a portion of their strain energy as heat (Tschoegl, 1989) during deformation. These devices are commonly used in buildings (Holmes, 1995) and aerospace structures. The dampers are more practical than auxiliary mass treatments due to their relative weight, size and mounting options. However, they are problematic in that their material behavior may deviate from the most common mathematical models, i.e., viscoelastic properties can be hard to model. Engineers idealize their behavior as elastic springs and viscous damping elements to simplify analysis.

1.2 Motivation

Published research on control algorithms and stability of the DR in many cases assumes that both the isolated DR and host structure behave as classic viscously damped systems (Olgac and Holm-Hansen, 1994; Olgac et al., 1997; Olgac and Hosek, 1997), however, the damping in some real-world systems is better represented by VE rather than viscous material models. In these systems, the VE properties may be purposely added or may reveal themselves due to temperature dependence.

There are a number of publications, notably Renzulli et al. (1999), Jalili and Olgac (2000), Jalili and Olgac (2000), and Hosek and Olgac (2002) that present methods to account for arbitrary unmodeled variations in the mechanical properties of the system. These methods will generally solve the problem of VE material in the system because the control algorithm will optimize itself to meet the need of the situation. The intent here is to gain a better understanding of the problem that is posed to a DR when it encounters viscoelastic behavior. Importantly, what does viscoelastic behavior do to the damping capability of the DR? How does the system’s stability outlook change? What changes in the DR algorithm are necessary to account strictly for viscoelastic behavior in the system?

This research incorporates the effects of viscoelasticity into the mathematical model of a DR system and estimates the effects of unmodeled viscoelasticity. Additionally, the mathematical modeling presented here is useful in designing DR systems when the VE damping mechanism is present either in the host or absorber structure. With the information presented here, the DR
designer can better understand when to take a modal route to model a DR system as opposed to the VE routes as illustrated by the top path and bottom path, respectively, in Fig. 1.4.

![Figure 1.4 Illustration of the modeling methods compared.](image)

1.3 Hypothesis and Approach to Thesis

The central hypothesis of this thesis is that both the accuracy of the stability outlooks and control algorithms for a DR may be improved if either the DR or the host structure exhibits viscoelastic behavior and these properties are modeled in more detail. The key steps in testing this hypothesis are as follows:

- Reproduce the results from Olgac and Holm-Hansen (1994) and Olgac et al. (1997). This provides a basic knowledge of DR design and a set of benchmark results for later comparison.
- Use well established methods by Ferry (1970) and Tschoegl (1989) to represent the time dependent constitutive relationship for VE material by a Prony series.
- Model the effect of temperature changes using the time-temperature superposition principle based on work by Williams et al. (1955).
- Combine VE modeling techniques with DR design techniques to create DR systems with VE damping mechanisms.
- Perform stability analysis on various combinations of viscous and viscoelastic hosts and DR structures to compare the models’ predictions.

Figure 1.5 maps the process of creating the test Cases that are presented in later chapters.
Start

Use Prony series to fit viscoelastic data. Compare fit function to validate method.

Use time-temperature superposition to shift fit function to represent viscoelastic material at temperature of interest. Compare shifted function to raw data to validate time-temperature superposition.

Model viscoelastic damping with MSM’s and coefficients determined by fitting viscoelastic data with the Prony series.

Include temperature dependence into MSM using $T$.

Show that the MSM degenerates to a viscous damper when the stiffness of the spring elements go to infinity.

Reproduce DR system from Olgac and Holmes-Hansen (1994).

Use design method from Olgac and Holme-Hansen (1994) with MSM models replacing viscous dampers to create a generalized viscoelastic DR system.

Create mixed test cases by replacing MSM’s with viscous dampers.

Figure 1.5 Chart of thesis objectives
1.4 Thesis Organization

This thesis is organized into five chapters. Chapter 2 discusses viscoelastic material properties and their modeling. Chapter 3 provides information on tuned mass vibration absorbers, delayed resonators with both viscous and viscoelastic properties, and the stability of systems that use the DR vibration absorber. Chapter 4 presents three test cases that validate the hypothesis. These test cases examine systems in which viscoelastic dampers are at various locations within the entire structure. Finally, Chapter 5 presents the conclusions from this study and suggests avenues for future study.
Viscoelastic (VE) materials exhibit time- and temperature-dependent creep as well as stress relaxation. Mathematical models of viscoelasticity require additional dynamic elements to capture the time- and temperature-dependence. Researchers commonly employ a phenomenological approach, i.e., a superposition of discrete, simple, time-dependent responses to produce close approximations to actual system behavior. Additionally, we incorporate temperature dependence into these models by considering materials that are thermorheologically simple. This allows the use of a shift function that varies the properties of the models dynamic elements with changes in temperature (Williams et al., 1955).

This chapter builds the foundation for modeling viscoelasticity. We will use this groundwork to model DR systems in later chapters. This chapter is organized into several sections. First is the introduction to the Maxwell Model of viscoelastic behavior, where the equations that define the viscoelastic modulus are derived from Hooke’s Law. We present time-temperature superposition as a method to incorporate the effects of temperature on the dynamic mechanical properties of viscoelastic materials. Next is a small section describing options available for adding temperature effects to a model. Finally, the chapter is concluded by determining viscoelastic coefficients for the example VE material ISD-112 from test data.

2.1 Maxwell Model of Viscoelastic Behavior

The generalized Maxwell model is used in this study to model linear VE behavior. It consists of a number of Maxwell elements connected in parallel. The parameters of each Maxwell element relate to the modulus of the material that it models through the Prony series expansion. Since this method of modeling is phenomenological, there is the possibility that loss mechanisms other than solely linear viscoelasticity are at play in the models that follow. This section describes the relationship of stress and strain in a VE material in terms of a viscoelastic modulus. The viscoelastic modulus is then translated into a Prony series representation that can easily be placed into a generalized Maxwell model for the material that is being modeled.
Figure 2.1 General N-parameter Maxwell model.

Figure 2.1 shows an N-Parameter Maxwell Model of a viscoelastic material. The stiffness and damping parameters $G$ and $\eta$ are adjusted so that the predicted response matches experimental data. We will begin by exploring the constitutive relationship of viscoelastic material exposed to step strains and then consider time dependent strains.

Applying a step stress produces creep until some equilibrium strain value is reached and relieving the stress on the system causes the system to quickly recoil and then slowly recover until equilibrium is reached. When exposed to a step in strain $\varepsilon(t)=\varepsilon_0$ at time $t=0$, the stress in a Maxwell element model initially “jumps” and then “relaxes” to some equilibrium value. Figure 2.2 illustrates both behaviors of the Maxwell element. The relationship between the time-dependent stress and the step strain is

$$\sigma(t) = G(t)\varepsilon_0, \quad (2.1)$$

where $G(t)$ is called the relaxation modulus. The relaxation of VE materials can be approximated by a Prony series expansion

$$\tilde{G}(t) = G_e + \sum_{i=1}^{N} G_i e^{-t/\rho_i}, \quad (2.2)$$

where $G_e$ is the equilibrium modulus, $G_i$ are relaxation moduli, $\eta_i$ are viscosities, and $\rho_i=\eta_i/G_i$ are time constants (Ward and Hadley, 1997; Tschoegl, 1989). Similarly, the relaxation modulus can also relate stress and an arbitrary time-dependent strain

$$\sigma(t) = \int_0^t \tilde{G}(t - \tau) \, d\varepsilon(t) = \int_0^t \tilde{G}(t - \tau) \frac{d\varepsilon(t)}{dt} \, dt \quad (2.3)$$
via the Boltzmann superposition principle. The Laplace transform of Eq. (4.1) is

$$\sigma(s) = \tilde{G}(s) [s \varepsilon(s)] = \left[ s \tilde{G}(s) \right] \varepsilon(s) = G(s) \varepsilon(s),$$

(2.4)

where $G(s)$ is the viscoelastic modulus in the Laplace domain. This function defines the real and imaginary response of the material that it models. Substituting the Laplace transform of the relaxation modulus from Eq. (2.2) into Eq. (2.4) yields the function that defines the viscoelastic modulus

$$G(s) = s \tilde{G}(s) = s \left[ \frac{G_e}{s} + \sum_{i=1}^{N} \frac{G_i}{s + 1/\rho_i} \right] = G_e + \sum_{i=1}^{N} \frac{s \rho_i G_i}{s \rho_i + 1}.$$

(2.5)

Tschoegl (1989) and Ward and Hadley (1997) showed that if the strain is harmonic with a frequency $\Omega$, the resulting stress will have both in-phase and out-of-phase components. These components are related to the strain amplitude by the storage modulus, $\Re[G(j\Omega)]$, and the loss modulus, $\Im[G(j\Omega)]$, respectively. Equation (2.5) can then be written as

$$G(j\Omega) = \left[ G_e + \sum_{i=1}^{N} \frac{\Omega^2 \rho_i^2 G_i}{\Omega^2 \rho_i^2 + 1} \right] + j \left[ \sum_{i=1}^{N} \frac{\Omega \rho_i G_i}{\Omega^2 \rho_i^2 + 1} \right] = \Re[G(j\Omega)] + j \Im[G(j\Omega)].$$

(2.6)
2.2 Time-Temperature Superposition

Temperature affects the mechanical properties of a viscoelastic material (VEM) in much the same way as frequency. Ferry (1970) accounted for the effects of temperature on the constitutive relationship of thermorheologically simple materials using the time-temperature superposition principle. This method uses a temperature-dependent shift function \( a_T \) to mathematically convert temperature variations to frequency variations. Applying the time-temperature shift function to storage modulus data has the effect of shifting the function horizontally as illustrated in Fig. 2.3.

![Figure 2.3 Shift in the relaxation modulus due to temperature effects.](image)

The temperature shift is dependent upon a reference temperature \( T_0 \), the actual temperature \( T \), and two material dependent parameters, \( C_1 \) and \( C_2 \). In our study, \( a_T \) is calculated using the Williams-Landel-Ferry (WLF) equation (Williams et al., 1955)

\[
\log a_T = \frac{C_1(T - T_0)}{(T - T_0) - C_2}.
\]  \hspace{1cm} (2.7)

The parameter \( T_0 \) is chosen arbitrarily as long as it lies within a reasonable range which varies based on the type of material. The parameters \( C_1 \) and \( C_2 \) are material and reference dependent parameters that must be determined experimentally (Williams et al., 1955; Ozupek and Becker, 1992).
In order to apply the temperature shift factor it must be related to the time variable of the modulus data or function. When considering the Prony series in the time domain, the time-temperature shift factor is introduced via a function called the reduced time, $t_r(t, T)$, that is defined by

$$t_r(t, T) = \int_0^t \frac{d\tau}{a_T[T(\tau)]}. \quad (2.8)$$

In practice, the time is replaced by $t_r(t, T)$ in the Prony series thereby allowing temperature variations to be seen as variations in time (Ozupek and Becker, 1992). This study is concerned with isothermal conditions at various temperatures so the integral in Eq. (2.8) becomes

$$t_r(t, T) = \frac{t}{a_T} \quad (2.9)$$

(Williams et al., 1955). The reduced time concept is also valid for frequencies, i.e., for isothermal conditions

$$\Omega_r = \Omega a_T, \quad (2.10)$$

where $\Omega_r$ is the reduced frequency.

The modulus data used in this study is based on harmonic test performed at a number of temperatures. Figures 2.4 and 2.5 show the experimental storage and loss modulus data; however, in order to make this data useful it must be written as a functional relationship. First, a master curve is created. This is a coherent curve composed of the scattered segments of data acquired at different temperatures that are adjusted to $T_0$ by multiplying the temperature shift factors of the data sets with their associated frequency coordinate of Eq. (2.10). This results in the master curves shown in Figs. 2.6 and 2.7 for the storage and loss moduli. The modulus data is the combined information from the master curves for the storage and loss moduli. A simultaneous curve fit of Figs. 2.6 and 2.7 with the Prony series creates a function representing the modulus data. This function represents the modulus data at $T_0$ and can be shifted to a desired temperature by replacing the frequency with the reduced frequency, so Eq. 2.6 becomes

$$G(j\Omega_r, T) = \left[ G_e + \sum_{i=1}^{N} \Omega^2_i \rho^2_i G_i \right] + j \left[ \sum_{i=1}^{N} \frac{\Omega_i \rho_i G_i}{\Omega^2_i \rho^2_i + 1} \right] = \Re[G(j\Omega_r, T)] + j \Im[G(j\Omega_r, T)] \quad (2.11)$$

2.2.1 WLF Equation Parameter Determination for ISD-112

In order to create a functional relationship for the viscoelastic modulus we must first determine the time-temperature shift function $a_T$. This requires finding the constants $C_1$, $C_2$, and $T_0$ for the WLF equation Eq. (2.3) from the raw ISD-112 data set. The raw data usually comes
Figure 2.4  Raw storage modulus data for ISD-112 used in this study for simulations.

Figure 2.5  Raw loss modulus data for ISD-112 used in this study for simulations.
Figure 2.6  Shifted storage modulus data for ISD-112 used in this study for simulations.

Figure 2.7  Shifted loss modulus data for ISD-112 use in this study for simulations.
in sets of modulus of data that vary with frequency; each set of data points is taken at a different temperature. The data can be combined at a common temperature by shifting the sets via time-temperature superposition. The raw data is tabular and contains the temperature shift factor $a_T$ at various discrete temperatures.

Determining the WLF parameters is a two-step process. First, we recognize that if $T = T_0$ then the value of the shift function is unity, i.e., $a_T = 1$. This allows the reference temperature $T_0$ to be found graphically by plotting the experimental values of $a_T$ versus temperature and locating the temperature that corresponds to shift function at one. The plot of $a_T$ versus temperature in Fig. 2.8 illustrates where this intersection is located. The second step is to fit the $a_T$ data with the WLF equation using the newly determined value of $T_0$. This yields the parameters $C_1$ and $C_2$ simultaneously. Figure 2.9 plots the raw $a_T$ data along with the curve fit. Table 2.1 summarizes the WLF parameters for ISD-112.

2.2.2 Comparison of Temperature Modeling Methods

There are two methods used to model the temperature dependence of the complex modulus of VE materials. Figure 2.10 discusses and compares these two methods: 1) fitting raw data at the temperature of interest and 2) fitting a master curve which is shifted to the temperature of interest.

The first method, Option #1, of fitting raw data can be done in two ways. One technique requires that the modulus data be fit each time the temperature changes. Unfortunately, the data may not be available at the temperature of interest and the values of must be interpolated. This is cumbersome and inefficient. A similar technique could be used where the master curve data is shifted to the temperature of interest using $a_T$ and then fit. This too can be inefficient since data must be fit every time temperature changes. The work presented in this thesis uses Option #2. In this method the master curve is fit and then the modulus function is determined and shifted to the temperature of interest.

Both methods are comparable and accurate when compared to raw data. Figure 2.10 contains several curves. The solid curve is the modulus function determined by fitting data for ISD-112 at $T_0 = 300.484K$ and the dotted curve represents the modulus function obtained by shifting using

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>300.484K</td>
</tr>
<tr>
<td>$C_1$</td>
<td>3.5890</td>
</tr>
<tr>
<td>$C_2$</td>
<td>76.8544K</td>
</tr>
</tbody>
</table>
Table 2.2 Initial values used in the fitting function algorithm and values after fitting routine has completed.

<table>
<thead>
<tr>
<th>element</th>
<th>Initial</th>
<th>Fitted</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$G_i$</td>
<td>$\rho_i$</td>
</tr>
<tr>
<td>1</td>
<td>5.955</td>
<td>$1.2566 \times 10^{-4}$</td>
</tr>
<tr>
<td>2</td>
<td>6.9089</td>
<td>$5.0265 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>0.5595</td>
<td>$2.58 \times 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>1.7129</td>
<td>$1.9 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>0.2329</td>
<td>0.5165</td>
</tr>
<tr>
<td>e</td>
<td>0.0044</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

the frequency variable in the Prony series by $a_T$ where $T = 288.4K$. Clearly these plots indicate that all methods to represent the storage moduli overlap. Furthermore, it is easy to show that same level of accuracy is obtained with all methods.

2.3 Determining $G_e$, $G_i$, and $\rho_i$ for ISD-112

The previous sections outlined the mathematical background of the viscoelastic modulus function. Given the modulus function it is relatively simple to construct a Maxwell model for ISD-112, a VEM manufactured by 3M that is often cited in scientific literature. How do we fit the data to the Prony Series? How are the values of $G_e$, $G_i$, determined? Commonly a least squares algorithm is employed to perform the curve fit to the storage and/or loss data. The fitting of either the storage or loss data will provide the desired information, but simultaneous fitting of both sets of data will increase the accuracy of the modulus function (Austin, 1998; Park, 2001). The least squares fit minimizes the error between the functions $\Re[G](\Omega_k)$ and $\Im[G](\Omega_k)$ defined in Eq. 2.6 and 2.11 and the experimental data $G'_k$ and $G''_k$ (Appendix B) via the error function defined by Tschoegl (1989)

$$\text{Error} = \sum_{k=1}^{N} \left[ \left( \frac{\Re[G](\Omega_k)}{G'_k} - 1 \right)^2 + \left( \frac{\Im[G](\Omega_k)}{G''_k} - 1 \right)^2 \right].$$

(2.12)

Figure 2.11 shows the ISD-112 dynamic test data and the curve fit. The data is fitted by a Prony series where $G_e \neq 0$ in series with four Maxwell elements. A MATLAB (The MathWorks, 2002) function fmincon in conjunction with a code derived from Austin (1998) is used minimize the error function. In an effort to produce a good fit, several initial guess values were chosen via trial and error. Table 2.2 lists these guesses along with their corresponding values from the fitting algorithm. and the values determined by the fitting algorithm are also listed in Table 2.2
Figure 2.8 Plot of $a_T$ versus $T$ used to determine $T_0$.

Figure 2.9 Plot of $a_T$ versus $T$ which is fit to determine $C_1$ and $C_2$.
The set of raw data, if available, at or near the temperature of interest can be fit to determine the modulus function coefficients for the material at that particular temperature. This can be repeated as temperature changes occur. But in order to get an accurate representation of the modulus as temperature changes data must be taken at a large number of temperatures or some type of interpolations must be used to cover temperatures between those where data is available.

**Option #2**

The set of raw data can be shifted to a common temperature, $T_0$, this data can then be fit to determine the modulus function coefficients at $T_0$. The modulus function can be shifted to the temperature of interest by applying the time-temperature shift function $a_T$ to the coefficients $G_i$ or $\rho_i$. Upon a change in temperature $a_T$ must be reevaluated which shifts the modulus function to the new temperature.

**Figure 2.10** A short discussion of two methods available to model temperature and a comparison of the results
Figure 2.11  Dynamic test data with fitted function overplotted.

Figure 2.12  Dynamic test data with fitted function overplotted.
CHAPTER 3
Modeling of Vibration Absorbers and the Delayed Resonator

Chapter 2 developed the foundations of modeling viscoelastic (VE) materials and their temperature dependence. Chapter 3 combines this groundwork with models of delayed resonator (DR) systems (Ferry, 1970; Findley et al., 1989; Tschoegls, 1989; Olgac and Holm-Hansen, 1994; Olgac et al., 1997; Olgac and Hosek, 1997; Park, 2001). This is essential in testing the hypothesis that both the stability outlooks and control algorithms developed for both the viscously-damped host and delayed resonator will benefit from VE modelling if either exhibits viscoelastic behavior.

3.1 Tuned-Mass Vibration Absorber and Tuned-Mass Damper

Chapter 1 briefly introduced some of the various types of vibration cancelation devices. The following sections will expand upon that cursory examination. The tuned-mass vibration absorber (TMVA), as Fig. 1.1 shows, is an undamped oscillator that can be attached to structure to cancel excitations. In theory, this device can completely cancel a sinusoidal excitation if it is tuned such that its natural frequency matches the frequency of excitation to be canceled. Springs tend to exhibit some inherent damping and nonlinearity and excitations, unfortunately, may stretch the spring beyond this linear region. Additional complexity in the system (Agnes, 1997) model can help increase the accuracy of the predicted system response. Similarly, viscoelasticity in systems can be modeled with more accuracy by creating more detailed system models.

The transfer function of an undamped single-degree-of-freedom system with a TMVA attached takes the following form

\[
\frac{X(s)}{F(s)} = \frac{s^2 + \omega_a^2}{(ms^2 + k + ka)(s^2 + \omega_a^2) - \omega_a^2 ka}. \tag{3.1}
\]

If the TMVA is tuned such that \( \omega_a = \Omega \), then \( \frac{X(s)}{F(s)} \to 0 \), implying that the host system does not respond to the excitation \( F(s) \) at \( s = j\Omega \).
While the TMVA can theoretically cancel a sinusoidal excitation we have established that it does not work well with physical systems due to damping. It has another major disadvantage; it adds an additional resonance peak to the system. This is alleviated by the addition of a viscous damper producing a tuned-mass-damper (TMD). The TMD is a passive device that cannot completely cancel an excitation but has the ability to attenuate a broad range of excitation frequencies.

3.2 The Delayed Resonator

The DR is essentially an semiactive TMD. Theoretically, both the TMVA and DR can cancel all vibrations in the primary structure, but the DR can compensate for varying excitation frequencies and damping in the absorber substructure. In the VA damping is undesirable because it degrades the effectiveness of the absorber. Conversely, damping is essential to DR in order to maintain system stability. Olgac and Holm-Hansen (1994) conclude that DRs are useful over an extended frequency range when compared to an equivalent TMVA.

The DR operates on the same principles as the TMVA: a pair of poles are placed on the imaginary axis at the excitation frequency as shown in Fig. 3.1. The DR, however, accomplishes this through a control algorithm that calculates gain $g_c$ and time-delay $\tau_c$. The control algorithm
makes the DR unique among vibration cancellation devices. It is what gives the device the ability to overcome the damping in the DR substructure and adjust for variable excitation frequency.

Figure 3.2 shows and isolated DR. Its equation of motion is

$$m_a \ddot{x}_a + c_a \dot{x}_a + k_a x_a + g_c x_a(t - \tau_c) = 0$$

(3.2)

where $g_c$ and $\tau_c$ denote the actuator gain and time-delay, respectively. The Laplace transform of Eq. (3.2) yields the characteristic equation

$$T(s) = m_a s^2 + c_a s + k_a + g_c e^{-\tau_c s} = 0.$$  

(3.3)

The left-hand side of this equation becomes the numerator of the system’s transfer function when it is attached to a host (Olgac and Holm-Hansen, 1994).

The DR control algorithm artificially maintains at least one pair of system poles on the imaginary axis of the complex plane at the driving frequency, $\Omega$. This is equivalent to forcing the numerator of the system transfer function to zero

$$TF = \frac{N(\Omega)}{D(\Omega)}.$$  

(3.4)

The algorithm originates from Eq. 3.3 by enforcing the condition

$$s = \pm j \Omega$$  

(3.5)

and solving for $g_c$ and $\tau_c$. Inserting Eq. 3.5 into Eq. 3.3 produces the complex equation

$$-m_a \Omega^2 + j c_a \Omega + k_a + g_c e^{-i\tau_c \Omega} = 0.$$  

(3.6)

![Figure 3.2 Schematic of an isolated delayed resonator](image-url)
Using Euler’s identity, Eq. 3.6 can be rewritten as

\[-m_a\Omega^2 + k_a + g_c \cos(\Omega \tau_c)] + j[c_a \Omega - g_c \sin(\Omega \tau_c)] = 0, \tag{3.7}\]

and this is only satisfied in general if

\[\Re : g_c \cos(\Omega \tau_c) = m_a\Omega - k_a \tag{3.8a}\]

\[\Im : g_c \sin(\Omega \tau_c) = c_a \Omega. \tag{3.8b}\]

Solving Eqs. (3.8) for \(g_c\) and \(\tau_c\) gives

\[g_c = \sqrt{(m_a\Omega^2 - k_a)^2 + (c_a\Omega)^2} \tag{3.9}\]

and

\[\tau_c = \frac{1}{\Omega} \left[\arctan \left(\frac{c_a\Omega}{m_a\Omega^2 - k_a}\right) + 2\pi \ell\right], \quad \ell = 0, 1, 2, 3, \ldots \tag{3.10}\]

The control parameter \(g_c\) is the actuator gain for the DR, and depending on the system it is proportional to the displacement, velocity (Filipovic and Olgac, 2002) or acceleration (Olgac et al., 1997; Olgac and Jalili, 1998; Jalili and Olgac, 1999) of the DR mass. The models in this thesis use a \(g_c\) that is proportional to the displacement. The time delay \(\tau_c\) directs the actuator to produce a control force that lags the displacement. It is obvious that the function which produces the time delay is transcendental because of \(2\pi \ell\) and \(\ell = 0, 1, 2, 3, \ldots\) implying the \(\tau_c\) can have an infinite number of values. As shown in Fig. 3.3, this causes the isolated DR characteristic equation to have an infinite number of poles.

### 3.3 Delayed Resonator With Viscoelastic Loss Mechanism

The response of a system with VE loss mechanisms is both temperature and frequency dependent. The central hypothesis of this thesis is that conventional methods of modeling viscous losses are inadequate when significant levels of VE behavior are present. Classical methods of modeling viscous loss neglect both temperature and frequency dependence. VE behavior is easily incorporated into the DR control algorithm by substituting the mathematics of the Generalized Maxwell model for stiffness and damping terms into characteristic equation which leads to

\[m_a s^2 + k_{ae} + \sum_{i=1}^{N} \frac{k_{ai} s}{(1/\rho_{ai}) + s} + g_{ve} e^{-\tau_{ve} s} = 0. \tag{3.11}\]
Following Olgac and Holm-Hansen (1994), Equation (3.11) is solved for $g_{ve}$ and $\tau_{ve}$ giving

$$g_{ve} = \sqrt{\left\{ \frac{m_a \Omega_r^2 - k_{ae}}{\Omega_r^2} - \sum_{i=1}^{N} k_{ai} \Omega_r^2 \right\}^2 + \left\{ \sum_{i=1}^{N} \frac{k_{ai}(1/\rho_{ai})\Omega_r}{\Omega_r^2 + (1/\rho_{ai}^2)} \right\}^2}$$

and

$$\tau_{ve} = \frac{1}{\Omega_r} \left\{ \arctan \left[ \frac{\sum_{i=1}^{N} k_{ai}(1/\rho_{ai})\Omega_r}{\sum_{i=1}^{N} \frac{k_{ai}\Omega_r^2}{\Omega_r^2 + (1/\rho_{ai}^2)}} \right] + \frac{2\pi l}{m_a \Omega_r^2 - k_{ae}} \right\}, \quad l = 0, 1, 2, 3, \ldots$$

(3.12)

(3.13)

DR control parameters that can adapt to variations in dynamic properties due to temperature and frequency change.

### 3.4 Stability Analysis

Optimal operation of DRs require that the device operate in a region of marginal stability. This requirement makes the stability of the global system a major concern when designing a DR for a particular host system. The time-delay in the control algorithm introduces an infinite number of poles to the global system. It is not an easy task to keep track of each of these poles individually. The location of poles in an optimal system are ultimately a function of excitation frequency. The diagrams in Fig. 3.4 show how the DR poles vary along the imaginary axis with changes in frequency while, the continuum of system poles vary along both the real and imaginary axis with frequency.
Fortunately, methods have been developed to assess the stability of systems where the location of each of poles is not readily available.

![Diagram of DR and system poles](image)

Figure 3.4 Comparison of the effect of frequency on DR poles and system poles.

Olgac and Hosek (1997) used the stability chart method to analyze the stability of systems that use the DR. The stability chart method provides a test to determine if any system poles have a positive real component, thus predicting instability. The method is based on the concept of D-subdivision (Filipovic and Olgac, 2002) which utilizes the continuity property of quasi-polynomials for variable parameters. This property asserts that there is at least one continuous path between the roots of two polynomials $p_1(s, \tau_1, g_1)$ and $p_2(s, \tau_2, g_2)$, for which two sets of parameters, $[\tau_1, g_1]$ and $[\tau_2, g_2]$, are associated. If the path does not include a purely imaginary root, $j\omega$, both polynomials have the same stability property (Filipovic and Olgac, 2002). The stability chart method identifies the boundary between a region of stable operation and a region of unstable operation. It allows one to visually determine where the system of interest is operating with respect to that boundary. The stability boundary is determined by solving for the purely imaginary system roots. The Boundary Crossing theorem (Filipovic and Olgac, 2002) states that if a stable polynomial is changing on an interval $\Omega$ and its roots do not cross the imaginary axis, then the polynomial is stable on the entire interval $\Omega$. This is used to define the stability of the system. Since the global system is inherently stable for $g = 0$, this is used as a reference point which designates the stable region. If the polynomial crosses into the unstable region, a pole or a set of conjugate poles must cross the imaginary axis making the global system unstable (Filipovic and Olgac, 2002).
To illustrate the method of stability chart an example based on that given by Olgac and Hosek (1997) is presented. Figure 3.5 shows a viscously damped single-degree-of-freedom system that is coupled to a viscously damped DR. The global system has equations of motion

\[ m_a \ddot{x}_a + c_a (\dot{x}_a - \dot{x}) + k_a (x_a - x) + g x_a (t - \tau) = 0 \] (3.14)

and

\[ m \ddot{x} + c \dot{x} + k x - c_a (\dot{x}_a - \dot{x}) - k_a (x_a - x) - g x_a (t - \tau) = f(t). \] (3.15)

The system transfer function is written as

\[ X(s) = \frac{T(s)}{E(s) + R(s) e^{-\tau s}} F(s) \] (3.16)

where

\[ E(s) = mm_a s^4 + (m c_a + m_a c + m_a c_a) s^3 + (m k_a + m_a k + m_a k_a + c c_a) s^2 + (c k_a + c_a k) s + k k_a, \] (3.17)

\[ R(s) = m s^2 + c s + k \] (3.18)

and \( T(s) \) is the characteristic equation of the DR given in Eq. 3.3. The characteristic equation of the entire system is then

\[ E(s) + R(s) e^{-\tau s} = 0. \] (3.19)

A system is marginally stable if a conjugate pair of poles resides on the imaginary axis, i.e., \( s = \pm j \Omega_b \), and there are no poles in the right half plane. Plugging this solution into the characteristic equation.

Figure 3.5 Delayed resonator attached to a single-degree-of-freedom primary system
given in Eq. (3.19) and solving for \( g_m \) gives

\[
g_m = \left\| \frac{E(s)}{R(s)} \right\|.
\]

(3.20)

This equation gives the values of the gain \( g_m \) for which the entire system is marginally stable for the corresponding frequency \( \Omega_b \). Solving for \( \tau_m \) gives

\[
\tau_m = \frac{1}{\Omega_b} \left\{ \pi(2p + 1) - \left\langle \frac{E(s)}{R(s)} \right\rangle \right\}, \quad p = 0, 1, 2, \ldots
\]

(3.21)

The values of the time-delay for which the entire system is marginally stable for the corresponding frequency \( \Omega_b \). A parametric plot of the \( g_m \) versus \( \tau_m \) produces a curve that is defined as the stability boundary for the entire system. Overplotting this curve with the DR gain \( g_c \) versus time-delay \( \tau_c \) given in Eqs. 3.9 and 3.10 produce a complete stability chart as displayed in Fig. 3.6 (Olgac and Hosek, 1997). The frequency \( \Omega_b \) is only a frequency parameter used to produce purely imaginary poles of the global system. It should be stressed that \( \Omega_b \) and the driving frequency \( \Omega \) are not related.

The usefulness of the stability chart to determine system stability is based on the interactions of three polynomials. The solid curve represents a polynomial that we have denoted as \( p_m(s, \tau_m, g_m) \), a set of unique combinations of \( \tau_m \) and \( g_m \) that place the global system in the state of marginal stability. The dashed curve represents a polynomial that we will call \( p_c(s, \tau_c, g_c) \), where
the parameters $\tau_c$ and $g_c$ represent the state of the system were the DR is operating as desired. We let the polynomial $p_0(s, \tau_0, g_0)$ represent a polynomial where $\tau_0 = 0$ and $g_0 = 0$, a state in which the system is inherently stable. The polynomial $p_m$ represents the stability boundary as defined above. The inherently stable polynomial $p_0$ amounts to a point at the intersection of the axes of $\tau$ and $g$. The location of $p_0$ allows for the region between the $\tau$ and $g$ axes, and $p_m$ to be specified as a region of stability. Therefore, the region that lies beyond the polynomial $p_m$ is a region of instability as allowed by the boundary crossing theorem. Optimal operation of the DR requires that the system state coincide with polynomial $p_c$. Therefore, the system’s operating point lies on an interval of $p_c$ that resides in the stable region of the stability chart.

This method of assessing stability can be extended to systems with VE elements by replacing the stiffness and damping terms of $g_m$ and $\tau_m$ with their viscoelastic equivalents. From Eqs. (3.20) and (3.21), $E(s)$ and $R(s)$ from the global system characteristic equation are the only system dependent values needed to calculate the stability limit. $E(s)$ consists of the sum of all terms in the system characteristic equation that are not multiples of $ge^{-\tau s}$, while $R(s)$ consists of a sum all of the parameters that are multiples of $ge^{-\tau s}$, such that the sum of all of these terms are $R(s)ge^{-\tau s}$.

The system was simulated in MATLAB using state-space models. This representation of the system simplified the calculation of the stability limits and standardized these calculations for all cases. The matrices $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}},$ and $\hat{\mathbf{D}}$ are used in the state-space representation of the host system while $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}},$ and $\hat{\mathbf{D}}$ are used in the state-space model of the absorber substructure (Each of these matrices are expanded in Appendix A). In calculating the stability limit for any case with the DR, $E(s)$ and $R(s)$ are defined by

$$R(s) = [\hat{\mathbf{C}}(s\hat{\mathbf{I}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}]^{-1}$$  \hspace{1cm} (3.22)

and

$$E(s) = D(s)R(s) + m_\alpha s^2(D(s) - m_\alpha s^2))$$  \hspace{1cm} (3.23)

where $D(s)$ is defined by

$$D(s) = [\hat{\mathbf{C}}(s\hat{\mathbf{I}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}]^{-1}.$$  \hspace{1cm} (3.24)

This method of assessing stability appears to be complicated to use. When applied it is very elegant, the stability of DR systems at any particular operating point can be accessed visually using the plots that are generated. Stability ranges can be can be assessed by noting intersection points of the stability limit curve and DR parameter curve and backing out the corresponding frequencies.
CHAPTER 4

Numerical Experiments and Results

The previous chapters have presented the hypothesis that motivated the work this thesis describes, presented the Delayed Resonator (DR) theory that was published by Olgac and Holm-Hansen (1994) as well as others in succeeding publications, and presented theory from multiple authors that facilitate modeling of viscoelastic (VE) materials. This chapter will highlight results collected from a number of numerical experiments that provide support for the hypothesis.

In many cases, the dynamics of a system are represented by equivalent viscous (EV) model. It is common for an engineer to obtain model parameters through a modal test as is illustrated in the top half of Fig. 4.1. This methodology can be inadequate when modeling systems with VE damping especially when excitation frequency and/or temperature changes. Therefore, modeling VE behavior with Maxwell elements as illustrated in the bottom half of Fig. 4.1 can produce results that do not agree with the EV results.

![Figure 4.1 Illustration of the modeling methods compared.](image)

This possible disagreement in modeling results leads to the two main points that will be emphasized in this chapter:

1. Unmodeled VE behavior alters the location of the stability bounds.
2. Unmodeled VE behavior degrades DR performance.
Point 1 will be illustrated by divergent stability charts as illustrated on the right hand side of Fig. 4.1 as well as area plots that will show how system stability varies with temperature and frequency, and point 2 will be illustrated with time responses that compare the vibration canceling effectiveness of the DR.

Figure 4.2 shows a typical stability chart that compares the stability outlook predicted by VE versus EV modeling. First an explanation of the curves must be given to provide context for the conclusions that are drawn from the points that have been highlighted. Each of the four curves are labeled:

![Stability chart](image)

**Figure 4.2** Stability chart of a system with VE damping in DR showing curves produced by VE and EV system models.

**Equivalent Viscous Control Parameters (EVCP)** — the control parameters that are fed into the DR for optimal performance based on an EV model of DR substructure.

**Viscoelastic Control Parameters (VECP)** — the control parameters that are fed into the DR for optimal performance based on a VE model of DR substructure.

**Equivalent Viscous System Stability (EVSS)** — the boundary between stable and unstable operation of the DR as predicted by the EV model of VE damping in the system.

**Viscoelastic System Stability (VESS)** — the boundary between stable and unstable operation of the DR as predicted by using Maxwell elements to model VE damping in the system.

The plot in Fig. 4.2 has four points highlighted. These points show what predictions a completely EV model (EVCP intersects EVSS at point A) could lead to, what actual behavior
might ensue based on using EV control parameters (EVCP intersects VESS at point \( B \) and \( C \)), and actual system behavior if a completely VE model is constructed and used (VECP intersects VESS at point \( D \)). It is shown that the predicted range of stable operation can be drastically different depending on the modeling method used.

![Diagram of different cases](image)

Figure 4.3  This diagram shows the different locations of VE damping in the simulations that where run.

In this study we have considered the effect of unmodeled viscoelasticity on stability bounds and the effect of unmodeled viscoelasticity on performance. All of the cases are illustrated schematically in Fig. 4.3.

4.1 Effect of Unmodeled Viscoelasticity on Stability Bounds

In this section we show that neglecting to model VE in a system can affect an engineer’s ability to accurately predict stability limits. Here it is assumed that the VE in the system modeled is approximated as EV so we only consider EVCP, EVSS and VESS from Fig. 4.2. Therefore, we will compare points \( A \), \( B \) and \( C \) on the EVCP as illustrated in Fig. 4.4. This EVCP curve is parameterized by frequency and because frequency is of interest in our analysis of stability this curve is simplified to the frequency line shown in Fig. 4.5. Figure 4.5 is redrawn in Fig. 4.6 to show that modeling a VE system as EV can lead to falsely predicted pockets of stability or instability. The statements in Fig. 4.6 are conditional because VE properties vary with temperature and frequency which leads to points \( \Omega_B \) and \( \Omega_C \) varying along the frequency line with respect to point \( \Omega_A \). When EVCPs are used for Cases 1 and 2 in this paper it is generally true that, because the VESS curve tends to move downwards with increases in temperature, as temperature increases \( \Omega_B \) increases and
$\text{Gain, } g$

$\text{Time-Delay, } t$

stable range predicted by VE model

only a lower bound predicted by EV model

EVCP

stable range predicted by VESS

lowest frequency excitation for stable system predicted using EV model

highest frequency excitation for stable system predicted using VE model

Figure 4.4 EVCP with intersection points $A$, $B$, and $C$ labeled.

Figure 4.5 Frequency bounds on stability when using EVCPs for the VEDR.

$\Omega_C$ decreases until points $\Omega_B$ and $\Omega_C$ intersect, beyond this point instability is always predicted, this point will be shown more clearly when specific Cases are discussed.

Figure 4.6 Frequency bounds on stability and their implications when using EVCPs for the VEDR.
4.2 Case 1: Viscoelastically Damped DR on Viscously Damped Host

Case 1 represents a VEDR attached to a viscously damped host structure. The plot in Fig. 4.15 expands upon Fig. 4.5 to show how temperature influences the system stability limits specified by points $\Omega_A$, $\Omega_B$ and $\Omega_C$. The frequency line from Fig. 4.5 is redrawn in Fig. 4.7 to make the connection to the statements that were made above when discussing the frequency line. First note that $\Omega_A$ follows a vertical line as temperature varies because the mathematics of the EV model is insensitive to temperature. Next points $\Omega_B$ and $\Omega_C$ follow the dashed curves that approach each other as temperature increases. The area between these curves correspond to the frequencies and temperature at which the system of Case 1 will have stable operation if EVCPs are used. The area to the right the curve $\Omega_C$ and the area between $\Omega_A$ and $\Omega_B$ when $\Omega_A < \Omega_B$ represents a range of frequencies and temperatures where EV modeling of the Case 1 system would falsely predict system stability. When $\Omega_A > \Omega_B$ the area between $\Omega_A$ and $\Omega_B$ represents a range of frequencies and temperatures where EV modeling falsely predicts that the system is unstable. If the temperature drifts higher than the point where $\Omega_B$ and $\Omega_C$ intersect the system will always be unstable because the whole of the EVCP curve would be inside of the unstable region.

![Figure 4.7](image-url)
4.3 Case 2: Viscoelastically Damped DR on Viscoelastically Damped Host

Case 2 represents a VEDR attached to a VE damped host structure. The plot in Fig. 4.8 is equivalent to the plot given in Fig. 4.7 to show how temperature influences the system stability limits specified by points for Case 1. The frequency line from Fig. 4.5 is redrawn in Fig. 4.8 to make the connection to the statements that were made above when discussing the frequency line. The critical value $\Omega_A$ follows a vertical line and $\Omega_B$ and $\Omega_C$ follow curves that intersect as temperature varies as they did in Case 1. The implications of points $\Omega_A$, $\Omega_B$ and $\Omega_C$ are the same as in Case 1. The important difference between Case 1 and Case 2 is that the system is more sensitive to temperature. Therefore, instability is guaranteed when using control parameters from the EVCP curve at with an even smaller increase in temperature above $T_0$.

4.4 Case 3: Viscously Damped DR on Viscoelastically Damped Host

In Case 3 the DR is damped viscously and therefore is well modeled with the standard spring and dashpot elements. We assume that control parameters are calculated correctly to optimally cancel vibrations in the host system. Because the host system does display VE, the stability of
the system is modeled with both a VE and EV model. In the case given here, there is no high frequency stability bound observed in our models regardless of the modeling method that was used. Therefore, point $\Omega_C$ goes away and the frequency line from Fig. 4.5 is revised for this case to that shown in Fig. 4.9. This frequency line shows that the EVSS and VESS curves both only predict low frequency stability limits. Figure 4.10 shows the implication using the EVSS to predict stability for

![Figure 4.9](image)

Figure 4.9 Frequency bounds on stability when using EVCPs for the VEDR.

Case 3 and as in Cases 1 and 2, there are pockets where the EVSS curve will falsely characterize the system stability situation. The variation of the stability limits with temperature are given in

![Figure 4.10](image)

Figure 4.10 Frequency bounds on stability and their implications when using EVCPs for the VEDR.

Fig. 4.11. This figure is similar to Figs. 4.7 and 4.8, but because of the absence of $\Omega_C$, the curve labeled $C$ is also absent. The plot in Fig. 4.11 shows that there is only a small deviation between the stability predictions when the VESS and EVSS stability limits are compared over the range of $T_0 \pm 40K$. There are only small regions where the EVSS falsely predicts stability or instability.

### 4.5 Effect of Unmodeled Viscoelasticity on Performance

In this section performance of the VEDR is discussed in terms of the control parameters used. In theory, the performance attained by the DR is solely dependent on producing the correct controller inputs. We will only consider Case 1 in this discussion because it has viscoelastic damping in the DR structure and is more simple to model than Case 2. Case 3 does not have VE damping in the DR structure and is therefore not relevant to the discussion. Although excluded from the discussion in this section, the behavior of the Case 2 system can sufficiently be explained by behavior of Case 1.
Figure 4.11 Variation of frequencies $\Omega_A$, $\Omega_B$, and $\Omega_C$ with temperature for Case 3.

The frequency response of an open-loop isolated DR is shown in Fig. 4.12. The four curves in this figure are as follows:

**VEM($T_0$)** The frequency response of the DR structure predicted by modeling the VE damping in the system with Maxwell elements as described in previous sections.

**Viscous Approx.** The frequency response of the DR structure as predicted by an EV model of VE damping.

**VEM($T_0 + 10K$)** The frequency response of the DR structure predicted using Maxwell models at $10K$ greater than $T_0$.

**VEM($T_0 - 10K$)** The frequency response of the DR structure predicted using Maxwell models at $10K$ less than $T_0$.

It is clear that the FRF curves in Fig. 4.12 are distinct and different, but they all represent the same DR structure. The **Viscous Approx.** curve is derived from the EV model of the viscoelasticity damped DR structure. The other three curves where created using Maxwell models to simulate the behavior of VEM. The curves labeled VEM($T_0 + 10K$) and VEM($T_0 - 10K$) show how the curve labeled VEM($T_0$) varies if the temperature is changed by $10K$. This plot clearly shows that the VE model is sensitive to temperature changes and that the EV model tends to work well near the resonant peak at $T_0$. In terms of performance, these modeling differences can play a role in how well...
the DR prevents vibration in the host structure, because modeling of the system is the basis of the control parameters that are calculated for the system.

It is generally true for the cases presented in this document that the gain required for the DR to operate optimally is inversely related to the frequency response $G(\Omega_j)$. This dependence is a result in the the fact that DR gain equation is composed of terms from the denominator of the DR structure’s FRF. It is shown by Olgac and Holm-Hansen (1994) that a local minimum in gain required by the DR occurs at the driving frequency of the peak value of the DR structure’s frequency response, namely in the Case of a viscously damped DR

$$\Omega_{\text{min}} = \omega_n \sqrt{1 - \zeta^2}. \quad (4.1)$$

Figure 4.13 is an adaptation of Fig. 4.2 and it just shows the curves for EVCP and VECP at 5K above $T_0$. The VECP curve is derived from the model that produced the VEM($T_0$) curve in Fig. 4.12 and the EVCP curve is derived from the model that produced the Viscous Approx. curve in Fig. 4.12. The plot in Fig. 4.12 shows that an increase in temperature tends to cause the peak of the FRF to increase and the gain to decrease when VE modeling is used with respect to the curves created by EV modeling. Figures 4.12 and 4.13 show that in some situations the EV and the VE models of the system can differ causing the optimal control parameters calculated from these models to disagree. It is not probable that both sets of control parameters will produce the optimal result that is sought.

The top plot ($H(j\Omega)$ vs. Frequency) in Fig. 4.14 shows that for Case 1 disagreement of EVCPs with VECPs does lead to non-optimal result when EVCPs are applied to the VE system
from Case 1. This is a FRF plot for the host structure of Case 1, with the DR attached. This figure actually shows four situations:

1. The response is out of range near zero if VECPs are used.

2. At temperature $T_0$ the use of the EVCPs produce a response in the host system as shown by the solid curve.

3. The dotted and dash-dot curves show the response of the system if EVCPs are used and the temperature decreases or increases by 10K, respectively.

Essentially, this Figure shows that for Case 1, applying the control parameters calculated from the EV model will hinder complete cancelation. Furthermore, the response of the host system is magnified if the temperature drifts away from $T_0$.

The bottom plot (% difference of Gain vs. Frequency) in Fig. 4.14 shows that the gain calculated by the EV model deviates from the gain calculated by the VE model at $T_0$, $T_0 + 10K$ and $T_0 - 10K$. This plot has been drawn with the frequency axis in line with the plot above and the line types of the curves matching to show correspondence. Most important is the behavior of the curves in these plots near the resonance frequency (near $10^{-1}$Hz). In the FRF plot, there is a dip in the solid curve near the resonant frequency, which corresponds to the percent difference approaching zero in the bottom plot. This is a result of an increase in accuracy of the EV model near the resonant frequency because the circle fit is actually applied at the resonant peak and is, therefore, most accurate there. When temperature drifts from $T_0$ the EV model does not approximate the
Figure 4.14  Top: Response of host system with VEDR modeled with a EV approximation at several temperatures. Bottom: Percent difference of gain calculated from a MSM vs. an EV model.
VE model as well and the gains deviate more sharply causing an amplification of the host system response as can be seen in the FRF plots at $T_0 + 10K$ and $T_0 - 10K$.

The discussion on how modeling of VEDR affects performance is finalized by providing time responses generated from simulated systems. Figure 4.15 gives plots from three scenarios involving Case 1. In each scenario, two host system time responses are given, the solid curve is the response if the VECPs are applied to the VEDR and the dash-dot curve is the response if EVCPs are applied to the VEDR. The middle plot was generated at temperature $T_0$ and the top and bottom plots were generated at temperatures $T_0 + 5K$ and $T_0 - 5K$ respectively. The plots in Fig. 4.15 mirror the FRF plots given in Fig. 4.14:

1. At each temperature shown, the steady-state time response goes to zero if the VECPs are used.
2. If the EVCPs are used at $T_0$ the host response is not canceled
3. Variation of temperature by $\pm 5K$ causes a significant increase in the host response amplitude if the EVCPs are used.

This chapter makes the two main points that in some cases the use of conventional modal methods to model VE behavior in DR systems can lead falsely predicted stability bounds and less than optimal system performance. Three example systems were simulated and there are a number of figures that show that for these Cases:

1. Conventional models that would be constructed from modal test vary from VE models.
2. Control parameters that are fed into the DR can deviate when different modeling methods are used.
3. The evaluation of system stability can be very significantly affected by the modeling method used.
4. Vibration canceling capability of the DR can suffer if conventional methods are used.
Comparison of Host System Responses $T_0=300.5K, T_1=305.5K$ and $f_c=11\, \text{Hz}$

Comparison of Host System Responses $T_0=300.5K, T_1=295.5K$ and $f_c=11\, \text{Hz}$

Host Response with $g_{\text{ve}}$ & $\tau_{\text{ve}}$
Host with $g_{\text{cir}}$ & $\tau_{\text{cir}}$

Figure 4.15  Response of the host system when DR is VE and temperature is changed.
CHAPTER 5

Summary, Conclusions and Future Work

The information presented in this thesis shows that the Maxwell and viscous approximation models of viscoelasticity in systems can cause DR performance to deviate appreciably. The first three chapters present theory that was used to create the system models and to assess stability. Chapter 4 presents numerical results of simulated experiments which provide support for the hypothesis.

Chapter 4 gives three cases and the steps taken to analyze these cases are given in the flowchart in Fig. 5.1. Case 1 is a SDOF viscously damped oscillator with a viscoelastically damped DR attached. Performance results are presented first in the form of a frequency response chart for the host system. This chart shows that if the DR control parameters are calculated from the viscoelastic (VE) model of the DR then the response of the host is zero because the exciting force is completely canceled. If the DR control parameters are calculated from a viscous approximation of the viscoelastic DR, the response of the host system is not completely canceled. Furthermore, if the temperature changes, the amplitude of the host’s response is increased.

Stability results for Case 1 are presented in the form of stability charts and stability range plots. These plots show, that if only a viscous approximation is used to model the VE DR, then the usefulness of the stability chart as a tool may suffer for two primary reasons. Most significantly, at temperature $T_0$, an upper stability limit occurs which the stability chart will not predict. Secondly, the stability range predicted by the viscous model of the system can vary significantly from the actual range of system stability if the temperature of the system changes.

The system in Case 2 only has viscoelasticity in the host structure. Viscous modeling fits the DR structure perfectly because it is viscous, therefore no performance analysis was performed. Stability analyses were performed and their results are given in the form of stability charts and stability range plots. Once again, these charts show that the stability of the system is affected by the method of modeling viscoelasticity in the system, the viscous approximation of damping in the system yields a different stability limit than the Maxwell model of viscoelasticity does. The effects of temperature also show up in this case through displacement of the stability limit produced by the Maxwell model.
Case 3 considers viscoelasticity in both the host and absorber sections. A performance analysis is not presented in this case since the DR is self-contained and performance of the DR is solely dependent on the ability to model the DR substructure. This has been done for Case 1. The stability situation of this system is considered. Stability charts and stability range plots are presented that show that significant modeling problems can occur if viscoelastic damping is modeled as approximately viscous in both substructures. Most significant is Fig. 4.7, which shows that if all of the VE damping in the system is approximated as viscous, then instability will be predicted above a certain temperature level at all frequencies shown on the plot. There is a separate set of curves in this figure that show that in actuality if the DR control parameters are calculated from a viscous approximation of damping in the system, the system will only be stable over a small frequency range at $T_0$. Furthermore, if the temperature increases by 5K the system will not be stable at any frequency.

The information discussed in this thesis leads to the following general conclusions that should be considered when creating a system model for a system with DR vibration control:

1. Viscous approximations of VE damping in DR systems can create significant deviations in system performance models and stability outlooks with respect to models that use the generally accepted Maxwell model of viscoelasticity with time-temperature superposition.

2. If an engineer suspects that their system behaves viscoelastically, the extra step of testing the system for viscoelasticity should be performed to minimize the opportunity for unexpected stability and performance problems to occur upon implementation.

3. If VE modeling of the system in necessary, this document provides a recipe for generating system models that can be used to create models that consider viscoelasticity. There are also a number of citations that can be used as reference material to create viscoelastic models to fit individual needs.

Incorporating viscoelastic modeling into DR systems that have VE damping can increase modeling accuracy. The frequency and temperature effects of VE behavior are considered which allows for increased model accuracy as well as increased control algorithm accuracy in a number applications that could possibly be problematic if viscoelasticity is modeled as viscous.

An extension of the information presented in this document would be to verify the usefulness by experimentation on real world examples. Successful result would lead to the next step of applying this reasoning to other behaviors, i.e., in this document a behavior that is deviant from the idealized viscously damped behavior was modeled by fitting a function to experimental data and that model was inserted into a control algorithm and stability analysis to adapt a DR for use with VE material. Therefore, future examples of adapting DR to perform optimally in the presence of other non-idealized behaviors should be studied. Finally, the addition of self-heating to the models above can
be used to show how normal operation a system with a DR can create cases of less than optimal performance and even system instability.

System 1 had VE damping in DR

- Modeled VE damping in DR with Maxwell Model and considered it a perfect representation of the VE in the system
- Approximated VE damping in DR with a viscous model derived from modal analysis of the DR’s FRF
- Created g and τ from both the Maxwell and viscous models of the DR

Stability chart compares the stability outlook that is predicted if a viscous approximation is used to the actual stability outlook that would be observed and it compares the stability outlook that would be observed if control parameters are calculated from the viscous approximation as opposed to using the Maxwell model of the VE damping in the DR

- Inserted g and τ from viscous approximation of DR into system with VE damping in DR modeled with Maxwell model and showed how the response magnitude

System 2 had VE damping in host

- Modeled VE damping in host with Maxwell model and considered it a perfect representation of the VE in the system
- Approximated VE damping in host with a viscous model derived from modal analysis of the host’s FRF
- Created stability chart for the system using both the Maxwell and viscous models of the VE in the host

Stability chart compares the stability outlook that is predicted if a viscous approximation is used to the actual stability outlook that would be observed VE damping in the DR

System 3 had VE damping in both host and DR

- Modeled VE damping in DR and host with Maxwell Model and considered it a perfect representation of the VE in the system
- Approximated VE damping in DR and host with a viscous model derived from modal analysis of the host’s and DR’s FRFs
- Created g and τ from both the Maxwell and viscous models of the DR

Stability chart compares the stability outlook that is predicted if a viscous approximation is used to the actual stability outlook that would be observed and it compares the stability outlook that would be observed if control parameters are calculated from the viscous approximation as opposed to using the Maxwell model of the VE damping in the DR

Figure 5.1 Flow chart shows steps taken and the information the conclusion presents
APPENDICES
Appendix A

State Matrices Used in Simulations and Analysis

This appendix is used to give the A and B state matrices that were used to model the VE systems that were considered in this thesis. First the matrices for the isolated, grounded viscoelastically damped host and DR are given. Sets of matrices for each arrangement viscously damped host/viscoelastically damped DR, viscoelastically damped host viscously damped DR, and viscoelastically damped host/viscoelastically damped DR.

Grounded Host with Viscoelastic Damper

\[
\tilde{A} = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
\frac{-k_i+k_{i+1}+\cdots+k_n}{m} & 0 & \frac{k_1}{m} & \cdots & \frac{k_n}{m} & 0 \\
\frac{k_1}{c_1} & 0 & -\frac{1}{\rho_1} & 0 & \cdots & 0 \\
\frac{k_2}{c_2} & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{k_n}{c_n} & 0 & 0 & \cdots & 0 & -\frac{1}{\rho_{N-1}} & 0 \\
\end{pmatrix}
\]  
(A.1)

\[
\tilde{B} = \begin{pmatrix}
0 \\
\frac{1}{m} \\
0 \\
\vdots \\
0 \\
\end{pmatrix}
\]  
(A.2)

\[
\tilde{u} = \left( f(t) \right)
\]  
(A.3)
Grounded DR with Viscoelastic Damper

\[ \hat{A} = \begin{pmatrix} 
0 & 1 & 0 & 0 & \cdots & 0 \\
-k_e + k_1 + \ldots + k_n/m & 0 & k_1/m & \ldots & k_n/m & 0 \\
-k_1/e_1 & 0 & -1/\rho_1 & 0 & \cdots & 0 \\
k_2/e_2 & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & -k_1/\rho_{n-1} & 0 \\
k_n/e_n & 0 & 0 & \cdots & 0 & -1/\rho_n 
\end{pmatrix} \]  
(A.4)

\[ \hat{B} = \begin{pmatrix} 
0 \\
0/\rho_1 \\
\vdots \\
0 \\
0 
\end{pmatrix} \]  
(A.5)

\[ \tilde{u} = \left( g x_1 (t - \tau) \right) \]  
(A.6)

Case 1: Simple-Viscoelastic System

\[ A = \begin{pmatrix} 
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
-k_e + k_1 + \ldots + k_n/m & 0 & k_1/m & \ldots & k_n/m & 0 \\
-k_1/e_1 & 0 & -1/\rho_1 & 0 & \cdots & 0 \\
k_2/e_2 & 0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & -k_1/\rho_{n-1} & 0 \\
k_n/e_n & 0 & 0 & \cdots & 0 & -1/\rho_n 
\end{pmatrix} \]  
(A.7)
\[ B = \begin{bmatrix} 0 & 0 & \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{m_a} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \end{bmatrix} \]  

(A.8)

\[ \vec{u} = \begin{bmatrix} f(t) \\ g x_3(t - \tau) \end{bmatrix}. \]  

(A.9)

Case 2: Viscoelastic-Simple System

\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{m} & \frac{1}{m} & k_1 & k_2 & k_3 & \cdots & k_{N-1} & k_N & \frac{c_a}{m} \\ \frac{k_1}{c_1} & 0 & -\frac{1}{\rho_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{k_2}{c_2} & 0 & 0 & -\frac{1}{\rho_2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \frac{k_{N-1}}{c_N} & 0 & 0 & \cdots & 0 & -\frac{1}{\rho_{N-1}} & 0 & 0 & 0 \\ \frac{k_N}{c_N} & \frac{c_a}{m_a} & 0 & \cdots & 0 & 0 & 0 & \frac{k_{N-1}}{m_a} & -\frac{c_a}{m_a} \\ \end{pmatrix} \]  

(A.10)

\[ B = \begin{pmatrix} 0 & \frac{1}{m} & \frac{1}{m} & 0 & 0 \\ 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \vdots \\ \end{pmatrix} \]  

(A.11)
\[ \tilde{u} = \begin{pmatrix} f(t) \\ g x_5 (t - \tau) \end{pmatrix} \]  \quad \text{(A.12)}

**Case 3: Viscoelastic-Viscoelastic System**

\[ A = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \]  \quad \text{(A.13)}

\[ B = \begin{pmatrix} 0 & 0 \\ \frac{1}{m} & \frac{1}{m} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & -\frac{1}{m_u} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \]  \quad \text{(A.14)}

\[ \tilde{u} = \begin{pmatrix} f(t) \\ g x_{(N+3)} (t - \tau) \end{pmatrix} \]  \quad \text{(A.15)}
where

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{\rho_1} & 0 & -\frac{1}{\rho_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -\frac{1}{\rho_{(N-1)}} \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

and

\[
A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{k_{au}+k_{a1}+\cdots+k_{an}}{m_a} & 0 & -\frac{k_{a1}}{m_a} & -\frac{k_{a2}}{m_a} & \cdots & -\frac{k_{an}}{m_a} \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

The system matrices where used in the MATLAB models that drove the simulations and plots that are the basis for the data presented above.
Appendix B

ISD-112 Raw Test Data

The information presented here is for the material ISD-112. It is a copy comments listed in the original code that was obtained for use in this research. There is also a table which shows a small snippet of the actual data.

1012 3M ISD 112 SHEAR AVAILABILITY: shelf, order DENSITY: FORM: tape roll TYPE:
Acrylic Transfer Tape TML: 1.20 CVCM: 0.27 WVR: 0.33 QUALITY: acceptable THICKNESS:
0.002, 0.005, 0.010 inches MANUFACTURER: 3M Industrial Specialties Division Building 220-7E-
01, 3M Center St. Paul, MN 55144-1000 612/733-1110 INFO:

Storage Requirements: Keep in cool, dry, and dark place. 3M publishes one year shelf life when stored at 70F, 50 percent R.H., and out of direct sunlight.

Shelf Life: Published as one year at above storage conditions.

Suggested Adhesives and Application Techniques: Self-adhesive. Degrease metals thoroughly before before application with trichloroethane, toluene or equivalent degreaser. Light wipe graphite epoxy substrates with degreaser until clean.

Enviromental Effects: unknown

Outgassing test performed by: Silicone Technology McGhan NuSil Corp. 1150 Mark Ave.
Carpinteria, CA 93013 805/684-8780

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Appendix C

Stability Charts

This appendix contains the stability charts that were omitted from the main text. Below you will find stability charts for Case 2 and Case 3. There are three charts for each case, these were produced at $T_0$, $T_0 + 5K$, and $T_0 - 5K$.

Figure C.1 Comparison of stability charts for system were VE damping in the Host is modeled with an $m$-$c_{eq}$-$k_{eq}$ approximation as opposed to an MSM, Case 2.
Figure C.2 Evolution of the stability chart in Fig. C.1 after temperature increase of 5K, Case 2.
Figure C.3  Evolution of the stability chart in Fig. C.1 after temperature increase of 5K, Case 2.
Figure C.4  Comparison of stability charts for system were VE damping in the Host and DR is modeled with an $m-c_{eq}-k_{eq}$ approximation as opposed to an MSM, Case 3.
Figure C.5  Evolution of the stability chart in Fig. C.4 after temperature increase of 5K, Case 3.
Figure C.6  Evolution of the stability chart in Fig. C.4 after temperature decrease of 5K, Case 3.
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